Exam 2, APPM 1360

Fall 2006

1) (a) $\int \frac{x^4}{x^2-1}\,dx = \int \left(\frac{x^4-1}{x^2-1} + \frac{1}{x^2-1} \right) dx$

$= \int \left[\frac{\cancel{(x^2-1)}(x^2+1)}{\cancel{x^2-1}} + \frac{1}{x^2-1} \right] dx$

$= \int \left[x^2+1 + \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1} \right] dx$

$= x^3/3 + x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$

$= x^3/3 + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$ //

(b) $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ Set $x = \sin u \Rightarrow dx = \cos u\,du$, then

$\int \frac{\sin^2 u}{\sqrt{1-\sin^2 u}} \cos u\,du = \int \frac{\sin^2 u}{\cancel{\cos u}} \cancel{\cos u}\,du = \int \sin^2 u\,du$

$= \int \frac{1}{2}(1-\cos 2u)\,du = \frac{u}{2} - \frac{1}{4}\sin 2u + C$

$x = \sin u \Rightarrow$

$u = \sin^{-1} x$

$= \frac{\sin^{-1} x}{2} - \frac{1}{2}\sin u \cos u + C = \frac{1}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + C$ //

2) The order of the numerator is smaller than the order of the denomenator. Thus look at the denomenator:

$\frac{1}{(2x-1)^3}$: one real root, repeated three times

$\frac{1}{x^2+9}$: no real roots, repeated once

$\frac{1}{x^2+x+1}$: no real roots, repeated twice

Hence

$$f(x) = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{(2x-1)^3} + \frac{Dx+E}{x^2+9}$$

$$+ \frac{Fx+G}{x^2+x+1} + \frac{Hx+I}{(x^2+x+1)^2}$$

//

3) (a) $\displaystyle\int_{-2}^{2} \frac{1}{x-1}\,dx = \lim_{\varepsilon \to 0^+} \int_{-2}^{\varepsilon+1} \frac{1}{x-1}\,dx \; +$

$$\lim_{\delta \to 0^-} \int_{\delta+1}^{2} \frac{1}{x-1}\,dx =$$

$$= \lim_{\varepsilon \to 0^+} \ln\left|\frac{-\varepsilon}{-3}\right| + \lim_{\delta \to 0^-} \ln\left|\frac{1}{\delta}\right| = \infty \;//$$

hence the integral diverges //

(b) $\displaystyle\int_{0}^{\infty} \frac{1}{\sqrt{x}\,(x+1)}\,dx$. Set $u = \sqrt{x} \Rightarrow 2u\,du = dx$

Then $\displaystyle\int_{0}^{\infty} \frac{2u\,du}{u(u^2+1)} = \int_{0}^{\infty} \frac{2\,du}{u^2+1} = \lim_{b \to \infty} \int_{0}^{b} \frac{2}{u^2+1}\,du$

$$= 2 \lim_{b \to \infty} \tan^{-1} u \Big|_0^b = \pi \;//$$

hence integral converges //

4) (a) $I_n = \int_0^1 (\ln x)^n dx = \int_0^1 \underbrace{\frac{d}{dx}(x)}_{dv} \underbrace{(\ln x)^n}_{u} dx$

Hence $u = (\ln x)^n$ $v = x$ and

$$I_n = \underbrace{x}_{v} \underbrace{(\ln x)^n}_{u} \Big|_0^1 - \int_0^1 \underbrace{x}_{v} \underbrace{n(\ln x)^{n-1} \frac{1}{x}}_{du} dx$$

$$= - \int_0^1 n (\ln x)^{n-1} dx = -n \int_0^1 (\ln x)^{n-1} dx$$

$$= -n I_{n-1} \Rightarrow$$

$$\boxed{I_n = -n I_{n-1}}$$

(b) The limit $\lim_{n\to\infty} [(-1)^n n!]$ does not exist because we get different values if n is odd and if n is even, hence the sequence diverges

5) (a) $a_n = \ln n - \ln(n+1) = \ln\left(\frac{n}{n+1}\right)$

as $n \to \infty$ $a_n = \ln\left(\frac{n}{n}\right) = \ln(1) = 0$

hence sequence converges //

(b) $a_n = \frac{\sin n}{\sqrt{n}}$, $|a_n| = \left|\frac{\sin n}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}}$

Choose $b_n = \frac{1}{\sqrt{n}}$, since $\lim_{n\to\infty} b_n = 0$, both

b_n and a_n will converge //

(c) As $n \to \infty$ $a_{2n} = 0$ and for the odd terms

$1 < a_1 < a_3 < a_5 \cdots < a_{2n-1} < a_{2n+1} < 2$

hence $\lim_{n\to\infty} a_{2n+1} = L$ where $1 < L < 2$

the $\lim_{n\to\infty} a_n$ doesn't exist hence the sequence diverges //

HOWEVER: Full credit will be given if you said since $a_n < 2$ it has an upper bound and it is a nondecreasing sequence, thus using the LUB theorem the sequence converges.

6) (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n+1}}$

$\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1 \neq 0$, hence the series diverges //

(b) Consider the integral $\displaystyle\int_2^{\infty} \frac{dx}{x \ln x^2} dx = \frac{1}{2}\int_0^{\infty} \frac{dx}{x \ln x}$

Set $u = \ln x \Rightarrow du = \frac{1}{x} dx$ and finally

$\displaystyle\frac{1}{2}\int^{\infty} \frac{du}{u}$ which diverges due to x^p theorem

hence the series diverges //

(c) Compare the series to $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{4n}\right)}{1/n} = \lim_{n\to\infty} \frac{\frac{\pi}{4}\left(\cancel{-\frac{1}{n^2}}\right)\cos\left(\frac{\pi}{4n}\right)}{\left(\cancel{-1/n^2}\right)}$$

$= \pi/4$. Hence since b_n diverges so does a_n //

7) Consider the sum

$$0.4 + 0.16 + 0.064 + 0.0256 + \cdots =$$

$$= 0.4\,[\underbrace{1 + 0.4 + 0.16 + 0.064 + \cdots}_{\text{geometric series with } r = 0.4}]$$

$$= 0.4 \cdot \sum_{n=0}^{\infty} (0.4)^n = 0.4\,\frac{1}{1-0.4} = \frac{0.4}{0.6} = \frac{2}{3}$$