Exam #3, APPM 1360

Fall 2006

(1) (a) Notice that $\lim_{n\to\infty} 2\cdot\cos(n\pi)$ doesn't exist since

$$\cos(n\pi) = (-1)^n = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$, hence the

sequence diverges //

(b) $a_n = \ln n + \ln(n+1) = \ln[n(n+1)] = \ln(n^2+n)$

as $n\to\infty$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln(n^2+n) = \infty$

hence sequence diverges //

(c) Since $a_n < a_{n+1}$ and $a_n > 0$, this is a nondecreasing sequence and since $a_{n+1} > 1$ it is bonded (has an upper bound), thus using the LUB theorem the sequence converges

2) (a) Consider the integral

$$\int_1^\infty \frac{x}{x^2+1}\,dx = \int_1^\infty \frac{1}{2}\,\frac{d(x^2+1)}{x^2+1} = \frac{1}{2}\ln(x^2+1)\Big|_1^\infty = \infty$$

hence the integral diverges and so does the series

Alternatively: Set $a_n = \frac{n}{n^2+1}$, $b_n = \frac{1}{n}$ and do the limit comparison test.

(b) Since $\lim_{n\to\infty} \ln(2n) = \lim_{n\to\infty} \ln(4n+2) = \infty$

The difference diverges.

Full credit will also be given for the following

$$\sum_{n=1}^\infty \ln(2n) - \sum_{n=1}^\infty \ln(4n+2) = \sum_{n=1}^\infty \left[\ln(2n) - \ln(4n+2)\right]$$

$$= \sum_{n=1}^\infty \ln\left(\frac{2n}{4n+2}\right)$$

Take $a_n = \ln\left(\frac{2n}{4n+2}\right)$ since $\lim_{n\to\infty} \ln\left(\frac{2n}{4n+2}\right) = \ln(1/2) \neq 0$

the series diverges

(c) Set $a_n = \sin\left(\frac{1}{2n}\right)$ and $b_n = \frac{1}{n}$ then by using the limit comparison test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/2n)}{1/n} = \lim_{n\to\infty} \frac{1/2(-1/n^2)\cos(1/2n)}{-1/n^2}$$

$$= 1/2$$

Hence since $b_n = \frac{1}{n}$ diverges so does a_n //

Note:

The n-term test gives you no information since

$$\lim_{n\to\infty} \sin\left(\frac{1}{2n}\right) = \lim_{n\to\infty} \sin(0) = 0$$

(d) Use the ratio test for $a_n = \frac{n!}{n^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}(n+1)}{\cancel{n!}} \frac{n^n}{(n+1)^{n+\cancel{1}}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

Write $\left(\frac{n}{n+1}\right)^n = \left[\frac{1}{n+1/n}\right]^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$

And since $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1$, hence the series converges //

Alternatively: $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} e^{n \ln\left(\frac{n}{n+1}\right)}$

$$= e^{\lim_{n\to\infty} n \ln(n/n+1)}$$

Then $\lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{\ln(n/n+1)}{1/n} = \lim_{n\to\infty} \frac{\frac{n+1}{n}\left(\frac{1}{n+1} - \frac{n}{(n+1)^2}\right)}{-1/n^2}$

$$= \lim_{n\to\infty} \left(-\frac{n}{n+1}\right) = -1$$

Hence $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$ //

3)(a) Using the ratio test we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{2n+2}}{n+1}\frac{n}{x^{2n}}\right| = \lim_{n\to\infty}\frac{n}{n+1}(x-2)^2$$

$$= (x-2)^2 < 1$$

Thus $|x-2|<1 \Rightarrow -1 < x-2 < 1 \Rightarrow 1 < x < 3$

Check the end points : $\underline{x=1}$: $\sum_{n=1}^{\infty}\frac{(1-2)^{2n}}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges

$\underline{x=3}$: $\sum_{n=1}^{\infty}\frac{(3-2)^{2n}}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Hence the interval of convergence is $1 < x < 3$ //

(b) Use again the ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{(-1)^{n+1}}{\sqrt{n+1}+1}(2x-3)^{n+1}\frac{\sqrt{n}+1}{(-1)^n}\frac{1}{(2x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\sqrt{n}+1}{\sqrt{n+1}+1}(2x-3)\right| = |2x-3| \leq 1$$

That means : $-1 < 2x-3 < 1 \Rightarrow \quad 2 < 2x < 4 \Rightarrow$

$1 < x < 2$. We need to check the end points

$\underline{x=1}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+1} (2-3)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$

this diverges due to p-test , $p = 1/2 < 1$

$\underline{x=2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+1} (4-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+1}$

Set $a_n = \dfrac{1}{\sqrt{n}+1}$, which will make this an alternating

series. Hence : $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}+1} = 0$

$a_n = \dfrac{1}{\sqrt{n}+1} > 0$ for all $n > 1$

$a_n > a_{n+1}$, since increasing the denominator of a fraction decreases the fraction. Hence the series

converges and finally $\quad 1 < x \leq 2$ //

4) (a) $f(x) = \sin x \;:\; f(\pi/4) = \sqrt{2}/2$

$f'(x) = \cos x \;:\; f'(\pi/4) = \sqrt{2}/2$

$f''(x) = -\sin x \;:\; f''(\pi/4) = -\sqrt{2}/2$

$f'''(x) = -\cos x \;:\; f'''(\pi/4) = -\sqrt{2}/2$

Hence using the power series formula

(keeping first four terms)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)}{n!}(x-\pi/4)^n = f(\pi/4) + f'(\pi/4)(x-\pi/4)$$

$$+ f''(\pi/4)\cdot\frac{1}{2}(x-\pi/4)^2 + f'''(\pi/4)\frac{1}{6}(x-\pi/4)^3 =$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x-\pi/4) - \frac{\sqrt{2}}{4}(x-\pi/4)^2 - \frac{\sqrt{2}}{12}(x-\pi/4)^3 \quad //$$

(b) $$\frac{x^2}{1+x} = x^2\cdot\frac{1}{1+x} = x^2 \underbrace{\sum_{n=0}^{\infty}(-x)^n}_{\text{geometric series}} = x^2\sum_{n=0}(-1)^n x^n$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{n+2} \quad //$$

Here $|x| < 1$

5) (a) We know that $\cos 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}$

Hence $\int_0^x \cos(2t)\,dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{(2t)^{2n}}{(2n)!}\,dt =$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int_0^x (2t)^{2n}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n} \int_0^x t^{2n}\,dt =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n} \frac{x^{2n+1}}{2n+1} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n)!(2n+1)}$$

(b) From part (a): $\int_0^{1/2} \cos(2x)\,dx = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!\,(2n+1)}$ (here $x = 1/2$)

This is an alternating series and in order to have 10^{-2} accuracy $|a_{n+1}| < 10^{-2} \Rightarrow \frac{1}{(2n+2)!} \frac{1}{(2n+5)} < 10^{-2}$

At $n = 0$, $a_1 = \frac{1}{2} \frac{1}{2} \frac{1}{5} > 10^{-2}$

At $n=1$ $a_2 = \frac{1}{2}\frac{1}{4!}\;\frac{1}{7} = \frac{1}{24}\;\frac{1}{7} < 10^{-2}$

Thus to 10^{-2} accuracy

$$\int_0^{1/2} \cos(2x)\,dx = \frac{1}{2}\left[1 - \frac{1}{2!\,3}\right] = \frac{1}{2}\left(1-\frac{1}{6}\right) = \frac{1}{2}\;\frac{5}{6}$$

$$= \frac{5}{12} \quad //$$

(c) Integrate $\int_0^{1/2} \cos(2x)\,dx = \frac{1}{2}\sin(2x)\Big|_0^{1/2} = 1/2\,\sin(1)$

$$= \frac{1}{2}\,0.841471 \approx 0.4207$$

Also $5/12 = 0.4167$

$$|0.4207 - 0.4167| = 0.004 < 10^{-2} \quad //$$

6)(a) We know that

$$\sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots$$

So that : $\sin(2x) + 1 - e^{2x} = \cancel{2x} - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \cdots$

$$\cancel{+1} - \cancel{1} - \cancel{2x} - \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} - \cdots$$

$$= -\frac{4x^2}{2} - 2\frac{(2x)^3}{3!} + \text{terms of } x^4 \text{ and higher}$$

Hence $\lim_{x \to 0} \dfrac{-\frac{4x^2}{2} - 2\frac{(2x)^3}{3!} + \text{terms of } x^4 \text{ and higher}}{x^2} =$

$$\lim_{x \to 0} \left[-2 - \frac{16}{6} x + \text{terms of } x^2 \text{ and higher} \right] =$$

$$= -2$$

(b) Using the definitions we have

$$\cosh x = \frac{1}{2}\left[e^{x}+e^{-x}\right] = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{x^{n}}{n!}+\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{n}}{n!}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}2\,\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n}}{(2n)!}$$

Hence

$$\lim_{x\to 0}\frac{1-1-\frac{x^2}{2!}-\frac{x^4}{4!}-\frac{x^6}{6!}-\cdots}{1-1+\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots} =$$

$$\lim_{x\to 0}\frac{-\frac{x^2}{2!}-\frac{x^4}{4!}-\frac{x^6}{6!}-\cdots}{\frac{x^2}{2}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots} = \lim_{x\to 0}\frac{\frac{\cancel{x^2}}{\cancel{2}}\left(-1-\frac{x^2}{12}-\frac{x^4}{320}-\cdots\right)}{\frac{\cancel{x^2}}{\cancel{2}}\left(1-\frac{x^2}{12}+\frac{x^6}{320}-\cdots\right)} =$$

$$\lim_{x\to 0}\frac{-1-\frac{x^2}{12}-\frac{x^4}{320}-\cdots}{1-\frac{x^2}{12}+\frac{x^4}{320}-\cdots} = \frac{-1}{1} = -1$$