Final Exam, APPM 1360

Fall 2006

1) (a) $\frac{d}{dx}\left[(2x)^{2x}\right] = \frac{d}{dx}\left[e^{2x\cdot\ln(2x)}\right] = \frac{d}{dx}e^u,$

$u = 2x\ln(2x)$

$$= e^u\cdot\frac{du}{dx} = e^{2x\ln(2x)}\frac{d}{dx}\left[2x\ln(2x)\right]$$

$$= (2x)^{2x}\cdot\left(2\ln(2x)+2\right)$$

$$= 2\left(\ln(2x)+1\right)\cdot(2x)^{2x} \quad //$$

(b) $\int \tanh(kx)\,dx = \int \frac{\sinh(kx)}{\cosh(kx)}\,dx = \int \frac{1}{k}\,\frac{d[\cosh(kx)]}{\cosh(kx)}$

$$= \frac{1}{k}\ln[\cosh(kx)] + C \quad //$$

2) (a) $\frac{df}{dx} + f = 2 \Rightarrow \frac{df}{dx} = 2 - f \Rightarrow \frac{df}{2-f} = dx \Rightarrow$

$\int \frac{df}{2-f} = \int dx = x + C \Rightarrow -\ln(2-f) = x + C \Rightarrow$

$2 - f = c \cdot e^{-x} \Rightarrow f = 2 - c e^{-x}$

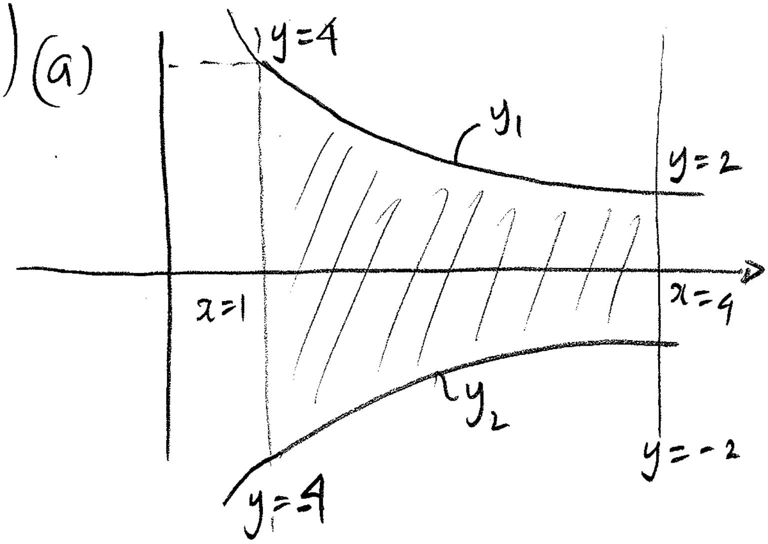
At $x = 0$, $f(0) = 0 \Rightarrow c = 2$

Finally : $f(x) = 2 - 2e^{-x}$ //

(b) $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(2 - 2e^{-x}\right) = 2 - 2 \lim_{x \to \infty} e^{-x}$

$= 2$ //

3) (a)



Total area:

$$A = \int_1^4 (y_1 - y_2)\,dx$$

$$= \int_1^4 \left(\frac{4}{\sqrt{x}} - \left(-\frac{4}{\sqrt{x}}\right)\right) dx$$

$$= 8 \int_1^4 \frac{1}{\sqrt{x}}\,dx \quad //$$

(b) Rotate around y-axis. Using shells we have

$$V = 2\pi \int_1^4 r \cdot h \cdot dx = 2\pi \int_1^4 x\,(y_1 - y_2)\,dx$$

$$= 2\pi \int_1^4 x \frac{8}{\sqrt{x}}\,dx = 16\pi \int_1^4 \sqrt{x}\,dx \quad //$$

(c) $$x = \frac{\int \tilde{x}\,dm}{\int dm} = \frac{\int_1^4 x(y_1 - y_2)\cdot\delta\cdot dx}{\int (y_1 - y_2)\cdot \delta\,dx} = \frac{\int_1^4 x^2 \cdot \frac{8}{\sqrt{x}}\,dx}{\int_1^4 x\,\frac{8}{\sqrt{x}}\,dx}$$

$$= \frac{\int_1^4 x^{3/2}\,dx}{\int_1^4 \sqrt{x}\,dx} \quad //$$

Due to symmetry $y = 0$ //

4) (a)(i) $\int_1^e x \ln x \, dx = \frac{x^2}{2} \cdot \ln x \Big|_1^e - \int_1^e \frac{x^2}{2} \frac{1}{x} \, dx$

$dv = x \Rightarrow v = x^2/2$

$u = \ln x$

$= \frac{e^2}{2} - \frac{1}{4} x^2 \Big|_1^e =$

$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4}$ //

(ii) $\int \frac{1}{x^2(x^2+1)} dx = \int \frac{x^2+1-x^2}{x^2(x^2+1)} dx = \int \frac{x^2+1}{x^2(x^2+1)} dx$

$- \int \frac{x^2}{x^2+1} dx$

$= \int \frac{1}{x^2} dx - \int \frac{1}{x^2+1} dx = -\frac{1}{x} - \tan^{-1} x + C$ //

(b) when $a > 0$ $x+a$ does not have any roots in $0 \le x < \infty$

hence $\int_0^\infty \frac{1}{(x+a)^4} dx = -\frac{1}{3} \frac{1}{(x+a)^3} \Big|_0^\infty = \frac{1}{3a^3}$ //

when $a \le 0$ $\int_0^\infty \frac{1}{(x+a)^4} dx = \lim_{c_1 \to -a^-} \int_0^{c_1} \frac{1}{(x+a)^4} dx + \lim_{c_2 \to -a^+} \int_{c_2}^\infty \frac{dx}{(x+a)^4}$

both integrals <u>diverge</u> since $\frac{1}{3} \frac{1}{(x+a)} \Big|_{x=-a} = \infty$

5) (a) $\sin(n\pi) \equiv 0$ for all integer n

hence sequence is identically zero //

(b) $a_n = \ln(2n+1) - \ln(n+1) = \ln\left(\frac{2n+1}{n+1}\right)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{2n+1}{n+1}\right) = \ln 2$, converges //

(c) $|a_n| \le \sin(1/n) \Rightarrow -\sin(1/n) \le a_n \le \sin(1/n)$

As $n \to \infty$ $\sin(1/n) = 0$ hence

$0 \le a_n \le 0 \Rightarrow \lim_{n\to\infty} a_n = 0$ by the

sandwich theorem, hence a_n converges. //

6) (a) Take integral test

$$\int_1^\infty \frac{x^2}{x^3+1}\,dx = \int_1^\infty \frac{1}{3}\frac{d(x^3+1)}{x^3+1} = \frac{1}{3}\ln(x^3+1)\Big|_1^\infty$$

$$= \infty$$

hence both series and integral diverge //

(b) $\lim\limits_{n\to\infty} \cos(1/2n) = \cos(0) = 1 \neq 0$ hence

series diverges //

(c) Take $a_n = \dfrac{(n!)^2}{(2n)!}$ and use ratio test

$$\frac{a_{n+1}}{a_n} = \frac{[(n+1)!]^2}{(n!)^2}\frac{(2n)!}{(2n+2)!} = \frac{\overbrace{(n+1)!}^{(n+1)\cancel{n!}}\cdot\overbrace{(n+1)!}^{(n+1)\cancel{n!}}}{\cancel{n!}\;\cancel{n!}}\;\frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= \frac{(n+1)^2}{(2n+2)(2n+1)} \quad \text{as } n\to\infty \quad \frac{a_{n+1}}{a_n} = \frac{1}{4} < 1$$

hence series converges //

7) Take again ratio test with $a_n = \frac{(-2)^n}{\sqrt{n}}(x+3)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-2)^{n+1}(x+3)^{n+1}}{(-2)^n(x+3)^n}\frac{\sqrt{n}}{\sqrt{n+1}}\right|$$

$$= \left|(-2)\cdot(x+3)\right| \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \quad \text{as } n \to \infty$$

$$\left|\frac{a_n}{a_{n+1}}\right| = 2\,|x+3| < 1 \Rightarrow -1/2 < x+3 < 1/2 \Rightarrow$$

$$-7/2 < x < -5/2$$

End points : $x = -7/2$: $a_n = \frac{(-2)^n}{\sqrt{n}}(3-7/2)^n = \frac{(-2)^n}{\sqrt{n}}\left(-\frac{1}{2}\right)^n$

$$= \frac{1}{\sqrt{n}} \text{, which diverges}$$

$x = -5/2$: $a_n = \frac{(-2)^n}{\sqrt{n}}(3-5/2)^n = \frac{(-2)^n}{\sqrt{n}}\frac{1}{2^n} = \frac{(-1)^n}{\sqrt{n}}$

This is an alternating series: (i) $a_n > 0$, (ii) obviously $\sqrt{n+1} > \sqrt{n} \Rightarrow a_n > a_{n+1}$ and (iii) $\lim_{n\to\infty} a_n = 0$ hence it converges. Finally $-7/2 < x \le -5/2$ //

8) (a) We know that $e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$

This is an alternating series with $a_n = (-1)^n \frac{x^n}{n!}$

Hence $\left| (-1)^{n+1} \frac{x^{n+1}}{(n+1)!} \right| < 10^{-2}$, $x = 0.2 = 2 \cdot 10^{-1}$

Then $\left| (-1)^{n+1} \frac{0.2^{n+1}}{(n+1)!} \right| = \begin{cases} \underline{n=0}: & 0.2 > 10^{-2} \\ \underline{n=1}: & \frac{0.2^2}{2!} = \frac{0.04}{2} = 0.02 > 10^{-2} \\ \underline{n=2}: & \frac{0.2^3}{3!} = \frac{8 \cdot 10^{-3}}{6} < 10^{-2} \end{cases}$

Hence $n = 2$ //

Finally: $e^{-0.2} = 1 - 0.2 + \frac{0.2^2}{2} = 0.8 + 0.02 = 0.82$ //

(Correct value $e^{-0.2} = 0.8187$ (!))

(b) This is a binomial series: $f(x) = \frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-1/2}$

Hence $f(x) = 1 + \sum_{n=1}^{\infty} \binom{-1/2}{n} (x^2)^n = 1 - \frac{x^2}{2} + \frac{3x^4}{8} - \frac{5x^6}{16} + \cdots$

For $f^{(4)}(0)$ look at the coefficient of x^4

$f^{(4)}(0) = \frac{3}{8} \cdot 4! = \frac{3}{\cancel{8}} \cancel{2} \cdot 3 \cdot \cancel{4} = 9$ //

a) (a) $\Delta = B^2 - 4AC = (2\sqrt{3}a)^2 - 4 \cdot 1(-1) = 12a^2 + 4 > 0$

for all a this is a hyperbola.

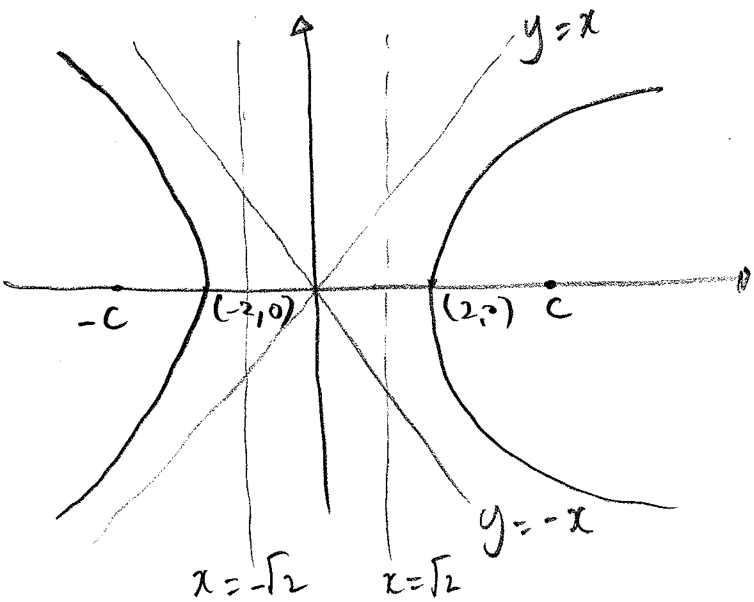
(b) To find the rotation we need to solve

$$\tan(2a) = \frac{B}{A-C} = \frac{2\sqrt{3}}{2} = \sqrt{3} \Rightarrow 2a = \pi/3 \Rightarrow$$

$a = \pi/6$ //. Then define $\begin{cases} x = x'\cos a - y'\sin a \\ y = x'\sin a + y'\cos a \end{cases}$

$\Rightarrow \begin{cases} x = \frac{1}{2}(x'\sqrt{3} - y') \\ y = \frac{1}{2}(x' + y'\sqrt{3}) \end{cases}$ to bring the equation in standard form //

(c) When $a = 0 \Rightarrow x^2 - y^2 = 4 \Rightarrow \frac{x^2}{4} - \frac{y^2}{4} = 1$



foci : $c = \pm\sqrt{a^2+b^2} =$

$= \pm\sqrt{4+4} = \pm 2\sqrt{2}$

eccentricity: $e = \frac{c}{a} = \sqrt{2}$

directrices : $x = \pm \frac{a}{e} = \pm\sqrt{2}$

10) (a) Recall $\left\{ \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} r^2 = x^2+y^2 \\ \tan\theta = y/x \end{array} \right.$

Hence $r = 2\sin\theta \Rightarrow r^2 = 2r\sin\theta \Rightarrow x^2+y^2 = 2y \Rightarrow$

$x^2+y^2-2y = 0 \Rightarrow x^2+y^2-2y+1 = 1 \Rightarrow x^2+(y-1)^2 = 1$ //

(b) We need to solve $1-\cos(2\theta) = 2\sin\theta$ and use symmetry for the second circle. $1-\cos 2\theta = 2\sin^2\theta$, thus

$2\sin^2\theta = 2\sin\theta \Rightarrow \left\{ \begin{array}{l} \sin\theta = 0 \\ \sin\theta = 1 \end{array} \right\}$ $\theta = 0, \pi/2$

$\theta = -\pi/2$ from symmetry //

(c) $A = 4\left[\frac{1}{2}\int_0^{\pi/2} (\text{circle})^2 - (\text{lemniscate})^2\right] = 2\int_0^{\pi/2} (4\sin^2\theta - 4\sin^4\theta)\,d\theta$

$= 8\int_0^{\pi/2} (\sin^2\theta - \sin^4\theta)\,d\theta$ //

(d) For the circle: $L = 4\int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = 4\int_0^{\pi/2} \sqrt{4\sin^2\theta + 4\cos^2\theta}\,d\theta$

$= 8\int_0^{\pi/2} d\theta = 4\pi$ //

For the lemniscate: $L = 4\int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$r = 2\sin^2\theta$

$\frac{dr}{d\theta} = 4\sin\theta\cos\theta$

$$= 4\int_0^{\pi/2} \sqrt{4\sin^4\theta + 16\sin^2\theta\cos^2\theta}\, d\theta$$

$$= 8\int_0^{\pi/2} \sin\theta\sqrt{\sin^2\theta + 4\cos^2\theta}\, d\theta$$

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Extra points: We need to show that the length of the lemniscate is smaller that the length of the circle (as it is obviously from the graph).

Either evalueate the integral in 10c or show that $r = 2\sin^2\theta$ for $0 \le \theta \le \pi/2$ reaches 1 faster than $r = 2\sin\theta$ or calculate $\frac{dy}{dx}$ for both cases and show that again $r = 2\sin^2\theta$ reaches 1 faster than $r = 2\sin\theta$.

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