Exam #1, APPM 1360
Spring 2007

(1)(a) $\int \tanh x \ln(\cosh x)\,dx = \int u\,du = u^2/2 + C =$

$u = \ln(\cosh x)$ $\qquad = \ln^2(\cosh x)/2 + C_1$

$du = \tanh x\,dx$

(b) $\int_1^2 \frac{x+2}{x^2+x}\,dx = \int_1^2 \left(\frac{x}{x^2+x} + \frac{2}{x^2+x}\right) dx = \int_1^2 \frac{1}{x+1}\,dx + 2\int_1^2 \frac{1}{x^2+x}\,dx$

$= \ln(x+1)\Big|_1^2 + 2\int_1^2 \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

$= \ln(x+1)\Big|_1^2 + 2\left(\ln x - \ln(x+1)\Big|_1^2\right) =$

$= 2\ln x\Big|_1^2 - \ln(x+1)\Big|_1^2 = \ln\left(x^2/x+1\right)\Big|_1^2 = \ln(8/3)$

(c) $\int (5-3x)e^{2x}\,dx = (5-3x)\frac{1}{2}e^{2x} - \int (-3)\frac{1}{2}e^{2x}\,dx =$

$u = 5-3x$

$v' = e^{2x} \Rightarrow v = \frac{1}{2}e^{2x}$ $\qquad = \frac{1}{2}(5-3x)e^{2x} + \frac{3}{4}e^{2x} + C_1$

(2)(a) $\underbrace{\cosh x \frac{dy}{dx} + \sinh x \, y}_{(\cosh x \, y)'} = e^{-x}$

$$(\cosh x \, y)' = \cosh x \, y' + (\cosh x)' y$$

$$= \cosh x \frac{dy}{dx} + \sinh x \, y$$

Hence $\frac{d}{dx}(\cosh x \, y) = e^{-x} \Rightarrow \cosh x \cdot y = -e^{-x} + C \Rightarrow$

$$y = \frac{C - e^{-x}}{\cosh x} \quad //$$

Initial condition: $y(0) = 0 \Rightarrow$

$$\frac{C-1}{1} = 0 \Rightarrow C = 1 \quad //$$

Finally $y = \dfrac{1 - e^{-x}}{\cosh x}$ //

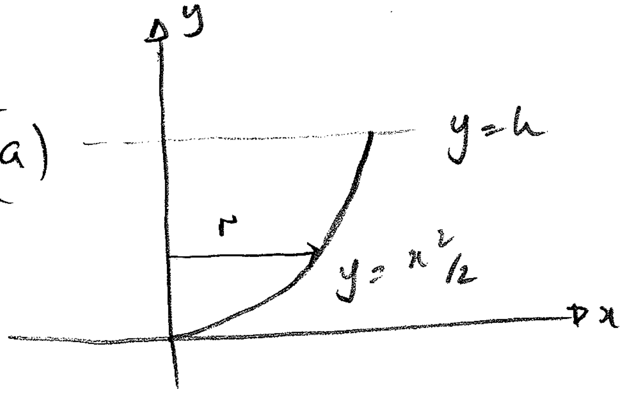
(b) $\cosh x \frac{dy}{dx} + \sinh x \, y = \cosh x \left[\frac{e^{-x}}{\cosh x} - \frac{1-e^{-x}}{\cosh^2 x} \sinh x \right]$

$$+ \sinh x \frac{1 - e^{-x}}{\cosh x} =$$

$$= e^{-x} - \frac{1-e^{-x}}{\cosh x} \sinh x + \frac{1 - e^{-x}}{\cosh x} \sinh x$$

$$= e^{-x} \quad //$$

(3) (a)



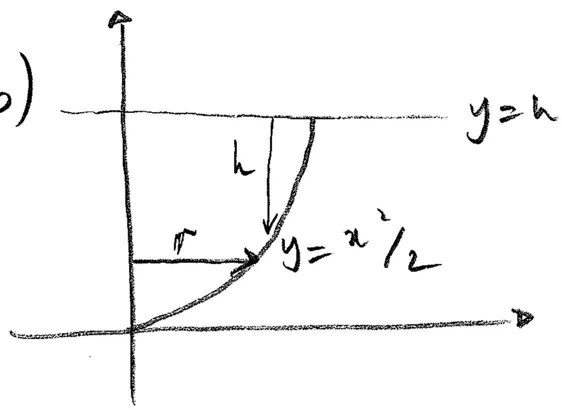
$$V = \int_0^h \pi r^2\,dy = \int_0^h \pi\left(\sqrt{2y}\right)^2 dy$$

$$= \pi \int_0^h 2y\,dy = 2\pi \left.\frac{y^2}{2}\right|_0^h$$

$$= \pi y^2\Big|_0^h = \pi h^2 \quad //$$

Since $V = 16\pi \Rightarrow \pi h^2 = 16\pi \Rightarrow h = 4$ //

(b)



$$V = \int_0^{\sqrt{2h}} 2\pi r h\,dx = \int_0^{\sqrt{2h}} 2\pi x\left(h - \frac{x^2}{2}\right)dx$$

$$= 2\pi \int_0^{\sqrt{8}} x\left(4 - \frac{x^2}{2}\right)dx$$

$$= 2\pi \int_0^{\sqrt{8}} \left(4x - \frac{x^3}{2}\right)dx =$$

$$= 2\pi\left[4\frac{x^2}{2} - \frac{x^4}{8}\right]_0^{\sqrt{8}} = 2\pi\left[16 - 8\right]$$

$$= 16\pi \quad //$$

(c)
$$S = \int_0^h 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_0^h 2\pi\sqrt{2y}\sqrt{1 + \frac{d}{dy}\left(\sqrt{2y}\right)}\,dy$$

$$= 2\pi \int_0^4 \sqrt{2y+1}\,dy = \frac{2\pi}{3}(2y+1)^{3/2}\Big|_0^4 = \frac{2\pi}{3}\left[27 - 1\right] = \frac{52\pi}{3} \quad //$$

(4)(a) $A = \int_{-1}^{1} (y_1 - y_2)\, dx = \int_{-1}^{1} \left(|x| + \sqrt{1-x^2} - |x| + \sqrt{1-x^2} \right) dx$

$= 2\int_{-1}^{1} \sqrt{1-x^2}\, dx = 4\int_{0}^{1} \sqrt{1-x^2}\, dx$, because of the symmetry //

Hence $4\int_0^1 \sqrt{1-x^2}\, dx = \pi \Rightarrow \int_0^1 \sqrt{1-x^2}\, dx = \pi/4$ //

(b) Since $\delta = \text{const}$ and the heart is symmetric w.r.t. x

$\bar{x} = 0$ // $\quad \bar{y} = \dfrac{\int_{-1}^{1} \tilde{y}\, dm}{\int_{-1}^{1} dm}$ $\qquad \tilde{y} = \frac{1}{2}(y_1 + y_2) = |x|$

$dm = \delta\, dA = \delta\,(y_1 - y_2)\, dx$

$= 2\sqrt{1-x^2}\, dx \cdot \delta$

Finally: $\bar{y} = \dfrac{\int_{-1}^{1} |x|\, \cancel{2} \sqrt{1-x^2}\, \cancel{\delta}\, dx}{\int_{-1}^{1} \cancel{2}\sqrt{1-x^2}\, \cancel{\delta}\, dx} = \dfrac{\cancel{2}\int_0^1 x\sqrt{1-x^2}\, dx}{\cancel{2}\int_0^1 \sqrt{1-x^2}\, dx}$

$= \dfrac{-1/3\,(1-x^2)^{3/2} \Big|_0^1}{\pi/4} = \dfrac{4}{3\pi}$ //

(c) Again using the symmetry

$$L = 2\int_0^1 \sqrt{1+\left(\frac{dy_1}{dx}\right)^2}\,dx + 2\int_0^1 \sqrt{1+\left(\frac{dy_2}{dx}\right)^2}\,dx$$

$$= 2\int_0^1 \sqrt{1+\left(1-\frac{x}{\sqrt{1-x^2}}\right)^2}\,dx + 2\int_0^1 \sqrt{1+\left(1+\frac{x}{\sqrt{1-x^2}}\right)^2}\,dx$$

$$= 2\int_0^1 \sqrt{1+1-\frac{2x}{\sqrt{1-x^2}}+\frac{x^2}{1-x^2}}\,dx + 2\int_0^1 \sqrt{1+1+\frac{2x}{\sqrt{1-x^2}}+\frac{x^2}{1-x^2}}\,dx$$

$$= 2\int_0^1 \sqrt{\frac{2-x^2}{1-x^2}-\frac{2x}{\sqrt{1-x^2}}}\,dx + 2\int_0^1 \sqrt{\frac{2-x^2}{1-x^2}+\frac{2x}{\sqrt{1-x^2}}}\,dx$$

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