Exam #2, APPM 1360

Spring 2007

(1) (a) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int \frac{\sec u \cdot \tan u}{\sec^2 u\sqrt{\sec^2 u-1}}\,du \quad (1+\tan^2 u=1)$

set $x = \sec u$

$dx = \sec u \tan u \, du$

$$= \int \frac{\cancel{\sec u} \cdot \cancel{\tan u}}{\sec^{\cancel{2}} u \cdot \cancel{\tan u}}\,du = \int \frac{1}{\sec u}\,du$$

$$= \int \cos u \, du = \sin u + C$$

However $x = \sec u \Rightarrow \cos u = \frac{1}{x}$ $\quad = \sqrt{1 - 1/x^2} + C$ //

$\Rightarrow \sin u = \sqrt{1-\cos^2 u} = \sqrt{1 - 1/x^2}$

(b) Define $a_n = \frac{\cos n}{n^2}$, since $\sum_{n=1}^{\infty} a_n$ converges

this is a telescopic series and thus

$$\sum_{n=1}^{\infty}\left\{\underbrace{\frac{\cos n}{n^2}}_{a_n} - \underbrace{\frac{\cos(n+1)}{(n+1)^2}}_{a_{n+1}}\right\} = a_1 = \frac{\cos 1}{1^2} = \cos 1 \quad //$$

(2) (a) $$\int_{-\infty}^{+\infty} \frac{1+x}{1+x^2}\,dx = \int_{-\infty}^{+\infty}\left(\frac{1}{1+x^2} + \frac{x}{1+x^2}\right)dx$$

$$= \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b \left(\frac{1}{1+x^2} + \frac{x}{1+x^2}\right)dx$$

$$= \lim_{\substack{a\to-\infty \\ b\to+\infty}} \left[\tan^{-1}x + \frac{1}{2}\ln(1+x^2)\right]_a^b$$

$$= \lim_{\substack{a\to-\infty \\ b\to+\infty}} \left[\tan^{-1}b - \tan^{-1}a + \frac{1}{2}\ln(1+b^2) - \tfrac{1}{2}\ln(1+a^2)\right]$$

which diverges since the ln-function diverges ($\ln\infty \to \infty$)

Hence the integral diverges //

(b) Since $b = -a = t$ the last line becomes

$$\lim_{t\to\infty}\left[\tan^{-1}t - \tan^{-1}(-t) + \frac{1}{2}\cancel{\ln(1+t^2)} - \frac{1}{2}\cancel{\ln(1+t^2)}\right]$$

$$= 2\lim_{t\to\infty}\tan^{-1}t = \pi$$ // (Cauchy principal value)

(c) These are <u>different</u> limits, <u>yes</u>, the two answers are consistent //

(3) (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} = \sqrt{2}$, converges //

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sinh(\ln n) = \sinh(\infty) = \infty$, diverges //

or $\sinh(\ln n) = \frac{1}{2}\left(e^{\ln n} - e^{-\ln n}\right) = \frac{1}{2}\left(n - 1/n\right)$

which also diverges

(c) $a_n = \frac{n!}{2^n} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 \cdots (n-1)\, n}{2\cdot 2\cdot 2\; 2\; 2\; 2 \cdots 2\cdot 2} \cdots$

$= \frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2}\cdot\frac{4}{2}\cdot\frac{5}{2}\cdots\frac{n-1}{2}\cdot\frac{n}{2}\cdots$

$\geq \frac{1}{2}\;\frac{2}{2}\;\frac{3}{2}\;\frac{n}{2} = \frac{3n}{2} = b_n$

However $\lim_{n\to\infty} b_n = \infty$ hence $\lim_{n\to\infty} a_n = \infty$, diverges //

OR from formula sheet

$\lim_{n\to\infty} \frac{x^n}{n!} \overset{x=2}{=} \lim_{n\to\infty} \frac{2^n}{n!} = 0 \Rightarrow \lim_{n\to\infty} \frac{n!}{2^n} = \frac{1}{0} = \infty$ //

(4)(a) If $a_n = \dfrac{n^2}{(2n+1)(2n+5)}$ $\quad \lim_{n \to \infty} a_n = \frac{1}{4} \neq 0$

hence series diverges //

(b) Using the integral test

$$\int_2^\infty \frac{1}{x \ln(x^2)}\, dx = \int_2^\infty \frac{1}{2x \ln x}\, dx = \int_{\ln 2}^\infty \frac{1}{2u}\, du$$

set $u = \ln x$
$du = \frac{1}{x} dx$

diverges due to "p"-test //

(c) Using the ratio test

$$\frac{a_{n+1}}{a_n} = \frac{(3n+3)!}{\cancel{(n+1)!}\cancel{(n+2)!}(n+3)!} \cdot \frac{n!\cancel{(n+1)!}\cancel{(n+2)!}}{(3n)!} = \frac{(3n+3)!}{(3n)!} \cdot \frac{n!}{(n+3)!}$$

$$= \frac{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}{\cancel{(3n)!}} \cdot \frac{\cancel{n!}}{(n+3)(n+2)(n+1)\cancel{n!}}$$

Hence $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 27 > 1$ and the series diverges //

(5) $\sum_{n=0}^{\infty} \frac{2^{3n}}{3^{2n}} = 1 + \frac{2^3}{3^2} + \frac{2^6}{3^4} + \cdots$

$= 1 + \frac{8}{9} + \frac{64}{81} + \cdots\cdots$ //

Set $r = \frac{2^3}{3^2}$ then $\sum_{n=0}^{\infty} \left(\frac{2^3}{3^2}\right)^n = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

since $r = \frac{8}{9} < 1$ this is a geometric series

and it converges //

$\sum_{n=0}^{\infty} \frac{2^{3n}}{3^{2n}} = \frac{1}{1-r} = \frac{1}{1-8/9} = 9$ //

(6) (a) Using the alternating series test

(i) $a_n = \frac{1}{n^{10}} > 0$, (ii) $\frac{1}{(n+1)^{10}} < \frac{1}{n^{10}} \Leftrightarrow a_{n+1} < a_n$

(iii) $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n^{10}} = 0$ hence the series

converges.

In addition $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^{10}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{10}}$ converges

thus the series converges absolutely //

(b) Using the reminder theorem $|R_N| < |a_{N+1}| < 4 \cdot 10^{-5}$

$\Rightarrow \frac{1}{(N+1)^{10}} < 4 \cdot 10^{-5} \Rightarrow (N+1)^{10} > \frac{1}{4} 10^5 = 25000$

- $\underline{N=1}$: $2^{10} = 1024 < 25000$
- $\underline{N=2}$: $3^{10} = 59049 > 2500$, hence we

need two $(N=2)$ terms

//

Extra credit :

$$\tau = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{F_n + F_{n-1}}{F_n} = 1 + \lim_{n\to\infty} \frac{F_{n-1}}{F_n}$$

$$= 1 + \lim_{n\to\infty} \frac{1}{F_n / F_{n-1}} = 1 + \frac{1}{\tau}$$

Finally : $\tau = 1 + \frac{1}{\tau} \Rightarrow \tau^2 = \tau + 1 \Rightarrow$

$$\boxed{\tau^2 - \tau - 1 = 0}$$

Thus $\tau = \dfrac{1 \pm \sqrt{5}}{2}$

However $\dfrac{F_{n+1}}{F_n} = 1 + \dfrac{F_{n-1}}{F_n} > 0$ for all $n \geq 4$

Hence $\tau = \frac{1}{2}(1 + \sqrt{5})$ //