Final Exam, APPM 1360

Spring 2007

(1) (a) $\cosh(\ln x) = \frac{1}{2}\left(e^{\ln x} - e^{-\ln x}\right) = \frac{1}{2}\left(x + \frac{1}{x}\right)$

Hence $\frac{d}{dx}\left[x \cosh(\ln x)\right] = \frac{d}{dx}\left[x \frac{1}{2}\left(x + \frac{1}{x}\right)\right]$

$$= \frac{1}{2}\frac{d}{dx}\left[x^2 + 1\right] = x \quad //$$

(b) $\int \frac{\sinh x}{1+\cosh x}\,dx = \int \frac{du}{u} = \ln u + C$

$$= \ln(1+\cosh x) + C \quad //$$

set $u = 1 + \cosh x$

$du = \sinh x \cdot dx$

(2) (a) Solve $\frac{dP}{dt} = -3\sqrt{P} \Rightarrow \frac{dP}{\sqrt{P}} = -3\,dt \Rightarrow$

$$\int \frac{dP}{\sqrt{P}} = \int (-3)\,dt \Rightarrow 2\sqrt{P} = -3t + C \Rightarrow$$

$$\sqrt{P} = -\frac{3}{2}t + C' \quad \text{or} \quad P = \left(-\frac{3}{2}t + C'\right)^2$$

when $t=0$, $P = 900$ thus $\sqrt{900} = C \Rightarrow$

$C = 30$

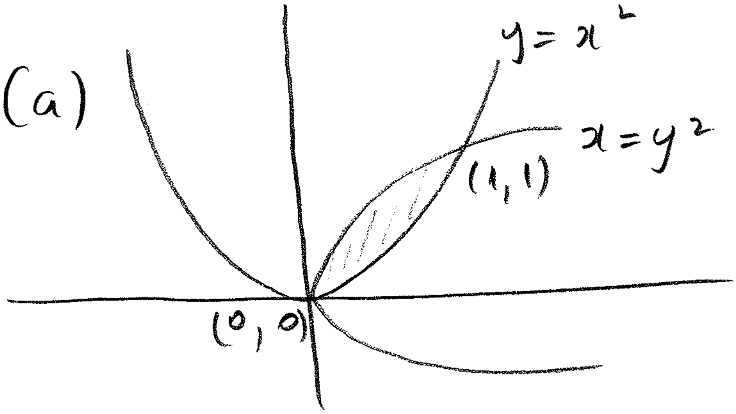
Finally $P = \left(-\frac{3}{2}t + 30\right)^2$ //

(b) when all fish vanish $P = 0 \Rightarrow$

$$\left(-\frac{3}{2}t + 30\right)^2 = 0 \Rightarrow -\frac{3}{2}t + 30 = 0 \Rightarrow$$

$t = 20$ weeks //

(3) (a)



$$\begin{Bmatrix} y = x^2 \\ x = y^2 \end{Bmatrix} \iff$$

$$\begin{Bmatrix} y = y^4 \\ x = y^2 \end{Bmatrix} \iff \begin{Bmatrix} y = 0,1 \\ x = 0,1 \end{Bmatrix}$$

$$\text{Area} = \int_0^1 (\sqrt{x} - x^2)\,dx = \int_0^1 (\sqrt{y} - y^2)\,dy \quad //$$

(b) <u>Washers</u>: $V = \pi \int_0^1 (R_{OUT}^2 - R_{IN}^2)\,dy$

$$= \pi \int_0^1 \left[(x_{OUT} - (-1))^2 - (x_{IN} - (-1))^2 \right] dy$$

$$= \pi \int_0^1 \left[(\sqrt{y} + 1)^2 - (y^2 + 1)^2 \right] dy \quad //$$

<u>Shells</u>: $V = 2\pi \int_0^1 r \cdot h\,dx$

$$= 2\pi \int_0^1 [x - (-1)] (\sqrt{x} - x^2)\,dx$$

$$= 2\pi \int_0^1 (x+1)(\sqrt{x} - x^2)\,dx \quad //$$

(c) $\text{Mass} = \int_0^1 \delta(x)\, dA = \int_0^1 c \cdot \sqrt{x}\,(\sqrt{x} - x^2)\, dx$

$$M_y = \int_0^1 \tilde{x}\, dM = \int_0^1 x\,\delta(x)\, dA = \int_0^1 x\, c \cdot \sqrt{x}\,(\sqrt{x} - x^2)\, dx$$

$$= \int_0^1 c\, x^{3/2}\,(\sqrt{x} - x^2)\, dx$$

$$M_x = \int_0^1 \tilde{y}\, dM = \int_0^1 \frac{1}{2}(\sqrt{x} + x^2)\,\delta(x)\, dA$$

$$= \int_0^1 \frac{1}{2}(\sqrt{x} + x^2)\, c\sqrt{x}\,(\sqrt{x} - x^2)\, dx$$

$$= \frac{1}{2}\int_0^1 c\sqrt{x}\,(x - x^4)\, dx$$

Finally: $\bar{x} = \dfrac{M_y}{M} = \dfrac{\int_0^1 x^{3/2}(\sqrt{x} - x^2)\, dx}{\int_0^1 \sqrt{x}\,(\sqrt{x} - x^2)\, dx}$ //

$$\bar{y} = \frac{M_x}{M} = \frac{\int_0^1 \sqrt{x}\,(x - x^4)\, dx}{\int_0^1 \sqrt{x}\,(\sqrt{x} - x^2)\, dx} \quad //$$

(4) (a) (i) $\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x+1)(x-1)}\,dx$

$$= \int \left(\frac{A}{x+1} + \frac{B}{x-1} \right) dx$$

Partial fractions: $\frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{x^2-1}$

$$= \frac{(A+B)x + (B-A)}{x^2-1} = \frac{1}{x^2-1}$$

Hence $\left\{ \begin{array}{l} A+B=0 \\ B-A=1 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} A=-B \\ 2B=1 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} A=-1/2 \\ B=1/2 \end{array} \right.$

Hence $\int \frac{1}{x^2-1}\,dx = \int \left(\frac{-1/2}{x+1} + \frac{1/2}{x-1} \right) dx$

$$= -\frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$$

$$= -\frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) + C$$

$$= \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + C \quad //$$

(ii) Set $x = \sec\theta \Rightarrow dx = \sec\theta \tan\theta \, d\theta$

$$x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$$

Finally: $\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{\sec\theta \tan\theta}{\tan^2\theta}\,d\theta$

$$= \int \frac{1}{\sin\theta}\,d\theta = -\ln\left(\frac{\cos\theta + 1}{\sin\theta}\right) + C$$

Since $x = \sec\theta = \dfrac{1}{\cos\theta} \Rightarrow \cos\theta = \dfrac{1}{x}$ and

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1 - 1/x^2}$$
$$= \sqrt{\frac{x^2-1}{x^2}}$$

Then

$$\int \frac{1}{x^2-1}\,dx = -\ln\left(\frac{1/x + 1}{\sqrt{\frac{x^2-1}{x^2}}}\right) + C = -\ln\left(\frac{x+1}{\sqrt{x^2-1}}\right) + C$$

$$= -\ln\left(\sqrt{\frac{x+1}{x-1}}\right) + C = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + C$$

(b) Use integration by parts:

$$\int_0^\infty x e^{ax}\, dx = x \frac{1}{a} e^{ax} \Big|_0^\infty - \int_0^\infty \frac{1}{a} e^{ax}\, dx$$

$u = x$

$v' = e^{ax}$

$\hookrightarrow v = \frac{1}{a} e^{ax}$

$$= \frac{1}{a} x e^{ax} \Big|_0^\infty - \frac{1}{a} \int_0^\infty e^{ax}\, dx$$

$$= \frac{1}{a} x e^{ax} - \frac{1}{a^2} e^{ax} \Big|_0^\infty$$

If $a = 1$: $(x-1)e^x \Big|_0^\infty \to \infty$, thus it diverges //

If $a = -1$: $-(x+1)e^{-x} \Big|_0^\infty = 1$, thus it converges //

Extra credit: If $a \geqslant 0$ then

$$\lim_{x \to \infty} x e^{ax} = \lim_{x \to \infty} e^{ax} = \infty$$

However when $a < 0$ both limits are zero

and the integral converges //

(5) (a) $\lim_{n\to\infty} \frac{1+(-1)^n}{n} = \begin{cases} \lim_{n\to\infty} \frac{2}{n} &, n \text{ is even} \\ \lim_{n\to\infty} \frac{0}{n} &, n \text{ is odd} \end{cases}$

$= 0$, for both cases

hence the sequence converges //

(b) $\lim_{n\to\infty} 2^{\cos(n\pi)} = 2^{\lim_{n\to\infty} \cos(n\pi)}$, which does not

exist hence the sequence diverges //

(c) $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{n\to\infty} \frac{1/n}{1/2\sqrt{n}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n}$

$= \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$, hence the

sequence converges //

(6) (a) Notice $\lim_{n \to \infty} \frac{n}{10n+17} = \frac{1}{10} \neq 0$

hence the series diverges //

(b) This is a geometric series with $r = \pi/4 < 1$

thus it converges //

(c) Take ratio test: $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| =$

$$= \lim_{n \to \infty} \frac{(n+3)!}{3^{n+1}\left[(n+1)!\right]^2} \frac{3^n (n!)^2}{(n+2)!} = \lim_{n \to \infty} \frac{3^n}{3^{n+1}} \frac{(n+3)!}{(n+2)!} \left(\frac{n!}{(n+1)!}\right)^2$$

$$= \lim_{n \to \infty} \frac{1}{3}(n+3) \frac{1}{(n+1)^2} = \frac{1}{3} \lim_{n \to \infty} \frac{n+3}{(n+1)^2} = 0 < 1$$

hence the series converges //

(7) Take the ratio test: $\lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| =$

$$= \lim_{n\to\infty} \left| \frac{(-4)^{n+1}}{\sqrt{2n+3}} x^{n+1} \frac{\sqrt{2n+1}}{(-4)^n} \frac{1}{x^n} \right| =$$

$$= 4|x| \lim_{n\to\infty} \frac{\sqrt{2n+1}}{\sqrt{2n+3}} = 4|x| < 1 \Rightarrow -\frac{1}{4} < x < \frac{1}{4} \; //$$

At $\underline{x = 1/4}$: $\sum\limits_{n=1}^{\infty} \frac{(-4)^n}{\sqrt{2n+1}} \frac{1}{4^n} = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$

This is an alternating series with $a_n = \frac{1}{\sqrt{2n+1}}$

and (i) $a_n > 0$, (ii) $a_{n+1} < a_n$, (iii) $\lim\limits_{n\to\infty} a_n = 0$

hence it converges.

At $\underline{x = -1/4}$: $\sum\limits_{n=1}^{\infty} \frac{(-4)^n}{\sqrt{2n+1}} \frac{1}{(-4)^n} = \sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}}$

which diverges because of "p"-test

Finally, the internal of convergence is

$$-1/4 < x \leq 1/4 \; //$$

(8) (a) We know that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Error $< |a_{n+1}| = \frac{x^{2n+2}}{(2n+2)!} < 10^{-2}$, since $x=1$

$$\frac{1}{(2n+2)!} < \frac{1}{100} \Rightarrow (2n+2)! > 100$$

when $n=2$, $6! > 100$, hence $n=2$ //

(b) $\cos(x) \approx \underset{n=0}{1} - \underset{n=1}{\frac{x^2}{2!}} + \underset{n=2}{\frac{x^4}{4!}}$ and hence

$$\cos(1) = 1 - \frac{1}{2!} + \frac{1}{4!} = 1 - \frac{1}{2} + \frac{1}{24} = \frac{24-12+1}{24}$$

$$= \frac{13}{24} \quad //$$

(c) $\sin x = -\frac{d}{dx}\cos x = -\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)x^{2n}}{(2n)!} = -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{d}{dx}x^{2n}$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}2n\,x^{2n-1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

//

(9) (a) Discriminant test : $B^2 - 4AC = 1 > 0$

hence this is a hyperbola //

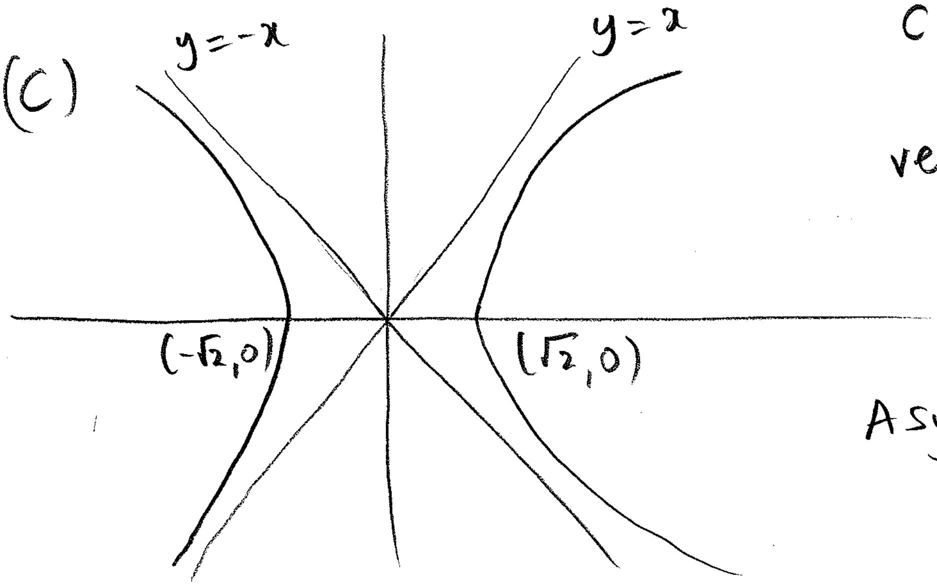
(b) $\tan 2\alpha = \dfrac{B}{A-C} = \dfrac{1}{0} = \infty \Rightarrow 2\alpha = \dfrac{\pi}{2} \Rightarrow \alpha = \pi/4$ //

$$\left.\begin{array}{l} x = x'\cos\alpha - y'\sin\alpha \\ y = x'\sin\alpha + y'\cos\alpha \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = \frac{1}{\sqrt{2}}(x'-y') \\ y = \frac{1}{\sqrt{2}}(x'+y') \end{array}\right.$$

Hence $x \cdot y = 1 \Rightarrow \frac{1}{\sqrt{2}}(x'-y')\frac{1}{\sqrt{2}}(x'+y') = 1 \Rightarrow$

$$\frac{1}{2}(x'^2 - y'^2) = 1 \Rightarrow \frac{x'^2}{2} - \frac{y'^2}{2} = 1 \quad //$$

(c)



$c = \sqrt{a^2+b^2} = \sqrt{2+2} = 2$

vertex $= \pm \dfrac{c}{a} = \pm \dfrac{2}{\sqrt{2}}$

$= \pm\sqrt{2}$ //

Asymptotes : $y = \pm x$ //

(10) (a) $\left\{ \begin{array}{l} x = r\cos\vartheta \\ y = r\sin\vartheta \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} r^2 = x^2 + y^2 \\ \tan\vartheta = y/x \end{array} \right\} \Rightarrow$

$\left\{ \begin{array}{l} r = \sqrt{x^2+y^2} \\ \vartheta = \tan^{-1}(y/x) \end{array} \right.$ and thus $r = \vartheta \Rightarrow$

$$\sqrt{x^2+y^2} = \tan^{-1}(y/x) \quad //$$

(b) When we cross the positive x-axis $y = 0$

Thus $y = r\sin\vartheta = \vartheta \cdot \sin\vartheta = 0 \Rightarrow \left\{ \begin{array}{l} \vartheta = 0 \\ \sin\vartheta = 0 \end{array} \right\}$

or $\vartheta = 2n\pi$, $n = 0, 1, 2, \ldots$ (recall positive x-axis)

Hence $P_1 = (0,0)$, $P_2 = (2\pi, 0)$, $P_3 = (4\pi, 0)$ //

(c) $\text{Area} = \int_0^{2\pi} \frac{1}{2} r^2 \, d\vartheta = \int_0^{2\pi} \frac{1}{2} \vartheta^2 \, d\vartheta = \frac{1}{2} \left. \vartheta^3/3 \right|_0^{2\pi} = \frac{4\pi^3}{3}$ //

(d) $\text{Length} = \int_0^{4\pi} \sqrt{r^2 + (r')^2} \, d\vartheta = \int_0^{4\pi} \sqrt{1 + \vartheta^2} \, d\vartheta$

$r = \vartheta \Rightarrow r' = 1$ //