Test 2 Review

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1) $y = \frac{\cos x}{x}$ $\quad y' = \frac{x(-\sin x) - \cos x}{x^2}$

1 $\quad x \cdot \left[\frac{x(-\sin x) - \cos x}{x^2}\right] + \frac{\cos x}{x} =$

$\frac{x^2}{x^2}(-\sin x) - \cancel{\frac{\cos x}{x}} + \cancel{\frac{\cos x}{x}} = -\sin x$ ✓

2 $\quad y(\pi/2) = \frac{\cos(\pi/2)}{x} = \frac{0}{\pi/2} = 0$ ✓

2) $y = \frac{1}{\sqrt{1+x^4}} \int_1^x \sqrt{1+t^4}\,dt$

Product Rule

$y' = -\frac{1}{2}\left[\frac{4x^3}{(\sqrt{1+x^4})^3}\right]\int_1^x \sqrt{1+t^4}\,dt$

$+ \frac{1}{\sqrt{1+x^4}}\left(\sqrt{1+x^4}\right)$

$y' = \left(\frac{-2x^3}{1+x^4}\right)\left(\frac{1}{\sqrt{1+x^4}}\int_1^x \sqrt{1+t^4}\,dt\right) + 1$

$y' = \left(\frac{-2x^3}{1+x^4}\right)y + 1 \Rightarrow y' + \frac{2x^3}{1+x^4} \cdot y = 1$

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3)

1) $\sqrt{x}\,\frac{dy}{dx} = e^{y+\sqrt{x}}$

$\sqrt{x}\,\frac{dy}{dx} = e^{y}e^{\sqrt{x}}$

$e^{-y}\,dy = \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$\int e^{-y}dy = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ let $u = \sqrt{x}$, $du = \frac{1}{2}\frac{1}{\sqrt{x}}dx$

$= 2\int e^{u}du$

$-e^{y} = 2e^{u} + C$

$-e^{y} = 2e^{\sqrt{x}} + C$

$$\boxed{e^{y} = -\frac{1}{2e^{\sqrt{x}}+C}}$$

2) $\frac{dy}{dx} - \frac{1}{x}y = 2\ln(x)$ $P(x) = -\frac{1}{x}$, $Q(x) = 2\ln x$

$\Rightarrow \int P(x)dx = -\int \frac{1}{x}dx = -\ln(x)$ $x>0$

$v(x) = e^{-\ln(x)} = \frac{1}{x}$

$y = x\int \frac{1}{x}(2\ln x)dx$

$= x\left[(\ln(x))^2 + c\right] = x(\ln(x))^2 + cx$

Solutions Review test 2 (3)

7.6 Improper integrals

$$\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta \qquad u=\theta^2+2\theta \quad du=2(\theta+1)\,d\theta$$

$$\int_0^3 \frac{1}{2\sqrt{u}}\,du = \lim_{b\to 0^+}\int_b^3 \frac{1}{2\sqrt{u}}\,du$$

$$= \lim_{b\to 0^+}\left[\sqrt{u}\right]_b^3 = \lim_{b\to 0^+}\left(\sqrt{3}-\sqrt{b}\right)$$

$$= \sqrt{3}-0=\sqrt{3}$$

32) $$\int_0^2 \frac{1}{|x-1|}\,dx = \int_0^1 \frac{1}{\sqrt{1-x}}\,dx + \int_1^2 \frac{1}{\sqrt{x-1}}\,dx$$

$$= \lim_{b\to 1^-}\left[-2\sqrt{1-x}\right]_0^b + \lim_{c\to 1^+}\left[2\sqrt{x-1}\right]\Big|_c^2$$

$$= \lim_{b\to 1^-}\left(-2\sqrt{1-b}\right)-\left(-2\sqrt{1-0}\right)+2\sqrt{2-1}$$

$$-\lim_{c\to 1^+} 2\sqrt{c-1} = 0+2+2+0=4$$

long problem skip it too short

$$\int_0^\infty \frac{1}{(x+1)(x^2+1)}\,dx$$

zero at $x=-1$ but not in interval
- use partial fractions

$$\lim_{b\to\infty}\left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1)+\frac{1}{2}\tan^{-1}x\right]_0^b$$

$$=\lim_{b\to\infty}\left[\frac{1}{2}\ln\left(\frac{x+1}{\sqrt{x^2+1}}\right)+\frac{1}{2}\tan^{-1}x\right]_0^b \Rightarrow$$

$$= \lim_{b \to \infty} \left[\frac{1}{2} \ln\left(\frac{b+1}{\sqrt{b^2+1}} \right) + \frac{1}{2} \tan^{-1} b \right]$$

$$- \left[\frac{1}{2} \ln \frac{1}{\sqrt{1}} + \frac{1}{2} \tan^{-1}(0) \right]$$

$$= \frac{1}{2} \ln(1) + \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \ln(1) + \frac{1}{2} \cdot 0 = \boxed{\frac{\pi}{4}}$$

Test 2 Review

Recursion

5a) $a_{n+2} = \dfrac{a_{n+1} + a_n}{2}$ $\quad a_0 = 3;\ a_1 = 13$

$$a_2 = \frac{13+3}{2} = \frac{16}{2} = 8$$

$$a_3 = \frac{13+8}{2} = \frac{21}{2}$$

b) $a_n = \dfrac{2^n - 1}{3n} \to 0$? I think

c) $a_n = n \tan\left(\dfrac{1}{n}\right)$

$$= n \frac{\sin(\frac{1}{n})}{\cos(\frac{1}{n})}$$

$$\Rightarrow \lim_{n\to\infty} n \frac{\sin(\frac{1}{n})}{\cos(\frac{1}{n})} = \cancel{\lim_{n\to\infty}}$$ sine functn

$$= \lim_{n\to\infty} n \sin\left(\tfrac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}}$$

$$= 1$$

$a_n = \ln(n) - \ln(n+1)$

$a_n = \ln\left(\frac{n}{n+1}\right) \Rightarrow$ convergesto zero

d) $a_n = \ln(n^2+3) - \ln(n)$

$$\Rightarrow \lim_{n\to\infty} \ln\left(\frac{n^2+3}{n}\right) = \ln\left[\lim_{n\to\infty} \frac{n^2+3}{n}\right]$$

$$= \ln[\infty] = \infty$$ - diverges

e) $a_n = \left(\frac{7}{n}\right)^{\frac{1}{n}}$

$$\lim_{n\to\infty} \left(\frac{7}{n}\right)^{\frac{1}{n}} = c$$

$$\lim_{n\to\infty} \ln\left(\left(\frac{7}{n}\right)^{1/n}\right) = \ln(c)$$

$$\lim_{n\to\infty} \frac{1}{n}\left[\ln(7) - \ln(n)\right] = \ln(c)$$

$$\lim_{n\to\infty} \frac{\ln(7)}{n} - \lim_{n\to\infty} \frac{\ln(n)}{n} = \ln(c)$$

ln -> ∞ slower than n

$0 = \ln(c)$

$e^0 = c \Rightarrow c = 1 \Rightarrow$ converges

Review Solutions

7.1

$$\int \frac{1}{\sqrt{2\theta-\theta^2}}\,d\theta$$

$$= \int \frac{1}{\sqrt{1-(\theta-1)^2}}\,d\theta \qquad \text{let } \begin{array}{l} u=\theta-1 \\ du=d\theta \end{array}$$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C$$

$$= \sin^{-1}(\theta-1) + C$$

$$\int 2x\,e^{x^2}\,dx \qquad \text{let } \begin{array}{l} u=\cancel{2}x^2 \\ du=2x\,dx \end{array}$$

$$= \int e^u\,du \Big|_{u=x^2} = e^{x^2} + C$$

7.2

$\int x^3 e^x dx$

let $dv = e^x\, dx$
$v = e^x$
$u = x^3$
$du = 3x^2\, dx$

$= x^3 e^x - \int 3x^2 e^x dx$

$dv = e^x$
$v = e^x$
$u = x^2$
$du = 2x\, dx$

$= x^3 e^x - 3\int x^2 e^x\, dx$

$= x^3 e^x - 3[e^x x^2 - 2\int x e^x dx]$

$= x^3 e^x - 3[e^x x^2 - 2[xe^x - \int e^x dx]]$

$dv = e^x$
$v = e^x$
$u = x$
$du = dx$

$= x^3 e^x - 3[e^x x^2 - 2[xe^x - e^x + c]]$

$= x^3 e^x - 3e^x x^2 + 6xe^x - 6e^x + 6C$

(10)

$$\int e^{\sqrt{3s+9}}\,ds \quad \text{let} \begin{bmatrix} 3s+9 = x^2 \\ ds = \frac{2}{3}x\,dx \end{bmatrix}$$

$$\rightarrow \int e^x \frac{2}{3}x\,dx = \frac{2}{3}\int xe^x\,dx \quad \text{let } dv = e^x$$

$v = e^x$

$u = x$

$du = dx$

$$= \frac{2}{3}\left(xe^x - \int e^x dx\right)$$

$$= 2/3\,(xe^x - e^x) + C$$

$$= 2/3\left(\sqrt{3s+9}\,e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$$

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$$\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx$$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{x^2+1}$$

$A = \frac{1}{2}$

$\cancel{B=0}$ $B = -1/2$

$C = 1/2$

$$\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx = \frac{1}{2}\int_0^1 \frac{1}{x+1}\,dx + \frac{1}{2}\int_0^1 \frac{(-x+1)}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln(|x+1|) - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x \Big]_0^1$$

$$= \frac{1}{2}\ln(2) - \frac{1}{4}\ln(2) + \frac{1}{2}\tan^{-1}1$$

$$-(\frac{1}{2}\ln(1) - \frac{1}{4}\ln(1) + \frac{1}{2}\tan^{-1}(0))$$

$$= \frac{1}{4}\ln(2) + 1/2\ \pi/4 = \boxed{\frac{\pi + 2\ln(2)}{8}}$$

$$\int \frac{z+1}{z^2(z-1)}\,dz \Rightarrow \frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1}$$

$$\Rightarrow z+1 = Az(z-1) + B(z-1) + Cz^2$$

$$\left.\begin{array}{l} A + C = 0 \\ -A + B = 1 \\ -B = 1 \end{array}\right\} \begin{array}{l} B = -1 \\ A = -2 \\ C = 2 \end{array}$$

$$\frac{z+1}{z^2(z-1)} = \frac{-2}{z} - \frac{1}{z^2} + \frac{2}{z-1}$$

$$\Rightarrow = -2\int \frac{1}{z}\,dz - \int \frac{1}{z^2}\,dz + 2\int \frac{1}{z-1}\,dz$$

$$\Rightarrow -2\ln|z| - \left[\frac{z^{-1}}{-1}\right] + 2\ln(|z+1|) + C$$

$$= -2\ln|z| + z^{-1} + 2\ln(|z-1|) + C$$

7.4
Grab Bag of Integrals

1) $\int \frac{\sqrt{y^2-25}}{y^3}\,dy \qquad y>5$

let $y = 5\sec\theta;\ 0<\theta<\pi/2$

$dy = 5\sec\theta\tan\theta$

$\sqrt{y^2-25} = 5\tan\theta$

$$\int \frac{\sqrt{y^2-25}}{y^3}\,dy = \int \frac{(5\tan\theta)(5\sec\theta\tan\theta)}{125\sec^3\theta}\,d\theta$$

$$= \frac{1}{5}\int \tan^2\theta\cos^2\theta\,d\theta = \frac{1}{5}\int \sin^2\theta\,d\theta$$

$$= \frac{1}{10}\int (1-\cos(2\theta))\,d\theta = \frac{1}{10}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right) - \left(\frac{\sqrt{y^2-25}}{5}\cdot\frac{5}{y}\right)\left(\frac{5}{y}\right)\right] + C$$

$$= \left[\frac{\sec^{-1}\left(\frac{y}{5}\right)}{10} - \frac{\sqrt{y^2-25}}{2y^2}\right] + C$$

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$$\int \frac{x^3}{x^2+4}\,dx$$

$x = 2\tan\theta \qquad -\pi/2 < \theta < \pi/2$

$dx = \frac{2}{\cos^2\theta}\,d\theta \qquad \sqrt{x^2+4} = \frac{2}{\cos\theta}$

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{(8\tan^3\theta)(\cos\theta)}{\cos^2\theta}\,d\theta$$

$$= 8\int \frac{\sin^3\theta}{\cos^4\theta}\,d\theta = 8\int \frac{(\cos^2\theta-1)(-\sin\theta)\,d\theta}{\cos^4\theta}$$

$t = \cos\theta$

$$\to 8\int \frac{t^2-1}{t^4}\,dt = 8\int \frac{1}{t^2} - \frac{1}{t^4}\,dt$$

$$= 8\left(-\frac{1}{t} + \frac{1}{3t^3}\right) + C = 8\left(-\sec\theta + \frac{\sec^3\theta}{3}\right) + C$$

$$= 8\left(-\frac{\sqrt{x^2+4}}{2} + \frac{(x^2+4)^{3/2}}{8\cdot 3}\right) + C$$

$$= \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$$