ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) APPM 1360, and (4) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and BOX IN YOUR FINAL ANSWERS. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, calculators, and electronics of any kind are NOT permitted.

1. (20 points) True or False: if the statement is *always* true, put **True** (not just T); if the statement is not *always* true, put **False** (not just F).

(a) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is defined to be $\lim_{b\to\infty} \int_{-b}^{b} \frac{dx}{1+x^2}$.

False: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is defined to be $\lim_{a\to-\infty} \int_{a}^{c} \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_{c}^{b} \frac{dx}{1+x^2}$ where c is any fixed number.

(b) If $a_{n+1} \geq a_n$ for all n, and a_n is bounded below, then $\lim_{n\to\infty} a_n$ exists.

False: If a_n increases it must be bounded above; if it decreases it must be bounded below.

(c) $\int_{1}^{\infty} \frac{dx}{x(\ln(x))^p}$ converges for $p > 1$.

False: Let $u = \ln(x)$, then the integral becomes $\int_{0}^{\infty} \frac{1}{u^p} du = \int_{0}^{1} \frac{1}{u^p} du + \int_{1}^{\infty} \frac{1}{u^p} du$, and the integral $\int_{0}^{1} \frac{1}{u^p} du$ diverges.

(d) $\int_{1}^{\infty} \frac{dx}{(x-2)^2}$ converges.

False: Let $u = x-2$, then $\int_{1}^{\infty} \frac{dx}{(x-2)^2} = \int_{-1}^{\infty} \frac{dx}{u^2} = \int_{-1}^{0} \frac{dx}{u^2} + \int_{0}^{1} \frac{dx}{u^2} + \int_{1}^{\infty} \frac{dx}{u^2}$, and the integrals with limits at 0 diverge.

2. (25 points) Determine if the following converge or diverge. Explain your reasoning.

(a) $\int_{\pi}^{\infty} \frac{2+\cos(x)}{x^{4/3}} dx$ Converges

$-1 \leq \cos(x) \leq 1,\ 1 \leq 2+\cos(x) \leq 3,\ \frac{1}{x^{4/3}} \leq \frac{2+\cos(x)}{x^{4/3}} \leq \frac{3}{x^{4/3}}$. The integral $\int_{\pi}^{\infty} \frac{3}{x^{4/3}} dx$ converges by the p-test, so by the DCT for integrals, $\int_{\pi}^{\infty} \frac{2+\cos(x)}{x^{4/3}} dx$ converges too.

(b)

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^4+16}} dx = \int_{-\infty}^{0} \frac{1}{\sqrt{x^4+16}} dx + \int_{0}^{\infty} \frac{1}{\sqrt{x^4+16}} dx$$

DCT: $x^4 < x^4+16,\ x^2 < \sqrt{x^4+16},\ \frac{1}{\sqrt{x^4+16}} < \frac{1}{x^2}$ To be exact, one cannot compare to $\frac{1}{x^2}$

LCT: $\lim_{x \to \pm\infty} \frac{\frac{1}{\sqrt{x^4+16}}}{\frac{1}{x^2}} = \lim_{x \to \pm\infty} \frac{x^2}{\sqrt{x^4+16}} = 1$ on the interval between 0 and $\pm\infty$, since the

Integrals $\int_{0}^{\pm\infty} \frac{1}{x^2} dx$ diverge. However, by splitting to three integrals $\int_{-\infty}^{-1} \frac{1}{\sqrt{x^4+16}} dx +$ $\int_{-1}^{1} \frac{1}{\sqrt{x^4+16}} dx + \int_{1}^{\infty} \frac{1}{\sqrt{x^4+16}} dx$ we may use the DCT or LCT with $\frac{1}{x^2}$ to prove the convergence of the first and last integrals. The middle integral is not improper, and therefore converges.

(c) $a_n = 8^{1/n}$ Converges to 1 via Table 8.1 (the formula sheet).

(d) $a_n = (\frac{n-2}{n})^n = (1-\frac{2}{n})^n$ which converges to e^{-2} via Table 8.1

(e) $a_n = \sin(n\pi - \frac{1}{n})$ As n becomes large, the argument of sine nears $n\pi$, and $\sin(n\pi) = 0$ (since n is an integer). Therefore the sequence converges.

3. a) 10 pts : $x^2 \frac{dy}{dx} = 2y, \; y(1) = e^{-2}$

$$\frac{dy}{dx} = \frac{2y}{x^2}, \quad \frac{dy}{y} = \frac{2dx}{x^2}$$

$$\int \frac{dy}{y} = \int \frac{2dx}{x^2}$$

$$\ln|y| = -\frac{2}{x} + C$$

$$y = e^{-\frac{2}{x} + C} = ke^{-\frac{2}{x}}, \quad k = e^C$$

$$y(1) = ke^{-\frac{2}{1}} = e^{-2} \quad \text{so} \quad k = 1$$

$$\boxed{y = e^{-2/x}}$$

3.b) 15 pts: $\frac{dy}{dx} + y = \frac{1}{4e^{2x}+1}$; $P(x) = 1$ $\quad Q(x) = \frac{1}{1+4e^{2x}}$

$$v(x) = e^{\int P(x)} = e^{x}$$

$$y(x) = \frac{1}{e^{x}} \int e^{x}\left(\frac{1}{1+4e^{2x}}\right) dx$$

$$u = 2e^{x}$$
$$du = 2e^{x}\,dx$$

$$y(x) = \frac{1}{e^{x}} \int \frac{\frac{1}{2}\,du}{1+u^{2}} = \frac{1}{2e^{x}}\left[\arctan(u) + c\right]$$

$$\boxed{y(x) = \frac{1}{2e^{x}}\left[\arctan(2e^{x}) + c\right]}$$

3. ~~a~~ c) 15 pts: $(t^2-3t+2)\frac{dx}{dt}=1, \quad x(3)=0$

$$\frac{dx}{dt} = \frac{1}{t^2-3t+2}$$

$$dx = \frac{dt}{t^2-3t+2} = \frac{dt}{(t-2)(t-1)}$$

$$\frac{1}{(t-2)(t-1)} = \frac{A}{t-2} + \frac{B}{t-1}; \quad 1 = A(t-1) + B(t-2)$$

@ $t=1$: $1 = -B$

@ $t=2$: $1 = A$

$$dx = \left[\frac{1}{t-2} - \frac{1}{t-1}\right]dt$$

$$x = \ln|t-2| - \ln|t-1| + C$$

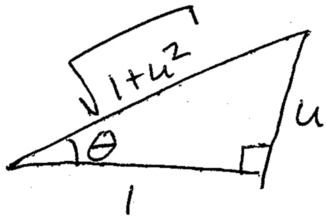
$$x(3) = \ln|1| - \ln|2| + C = \cancel{\ln|2|}\,0$$

$$C = \ln|2|$$

$$\boxed{x(t) = \ln|t-2| - \ln|t-1| + \ln|2|}$$

4) a) 15 points:

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}} = \begin{Bmatrix} u = e^t \\ du = e^t dt \end{Bmatrix} \int_{3/4}^{4/3} \frac{du}{(1+u^2)^{3/2}}$$



$$\sqrt{1+u^2} = \sec\theta$$
$$(1+u^2)^{3/2} = \sec^3\theta$$
$$u = \tan\theta$$
$$du = \sec^2\theta\, d\theta$$

$$\int_{3/4}^{4/3} \frac{du}{(1+u^2)^{3/2}} = \int_{u=3/4}^{u=4/3} \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int_{u=3/4}^{u=4/3} \cos\theta\, d\theta = \Big[\sin\theta\Big]_{u=3/4}^{u=4/3} =$$

$$= \left[\frac{u}{\sqrt{1+u^2}}\right]_{3/4}^{4/3} = \left[\frac{4/3}{\sqrt{1+\frac{16}{9}}} - \frac{3/4}{\sqrt{1+\frac{9}{16}}}\right] = \left[\frac{4/3}{\sqrt{\frac{25}{9}}} - \frac{3/4}{\sqrt{\frac{25}{16}}}\right] = \frac{4}{5} - \frac{3}{5} = \boxed{\frac{1}{5}}$$

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4) b) 15 points

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{-(x-1)}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x}} + \lim_{a\to 1^+} \int_a^2 \frac{dx}{\sqrt{x-1}}$$

$$\begin{Bmatrix} u = 1-x, & v = x-1 \\ du = -dx, & dv = dx \end{Bmatrix} = \lim_{b\to 1^-} \int_{x=0}^{x=b} \frac{-du}{u^{1/2}} + \lim_{a\to 1^+} \int_{x=a}^{x=2} \frac{dv}{v^{1/2}} =$$

$$= \lim_{b\to 1^-} \Big[-2u^{1/2}\Big]_{x=0}^{x=b} + \lim_{a\to 1^+} \Big[2v^{1/2}\Big]_{x=a}^{x=2}$$

$$= \lim_{b\to 1^-} \left[-2(1-b) + 2(1-0)\right] + \lim_{a\to 1^+} \left[2(2-1) - 2(a-1)\right]$$

$$= \left[-2(0) + 2\right] + \left[2 - 2(0)\right] = \boxed{4}$$

5) $\displaystyle\int_0^T e^{-x^2}dx = \frac{\sqrt{\pi}}{2} - \int_T^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} - \int_T^\infty \frac{(-2xe^{-x^2})dx}{(-2x)}$

$$\int_T^\infty \frac{-2xe^{-x^2}dx}{-2x} = \left\{ \begin{array}{ll} u = \frac{1}{2x} & v = -e^{-x^2} \\ du = \frac{-1}{2x^2}dx & dv = 2xe^{-x^2}dx \end{array} \right\}$$

$$= \left[\frac{-e^{-x^2}}{2x} - \int \frac{e^{-x^2}}{2x^2}dx \right]_T^\infty$$

$$= \left[0 + \frac{e^{-T^2}}{2T} - \underbrace{\int_T^\infty \frac{e^{-x^2}}{2\cancel{x}^2}dx}_{\text{assume this is small}} \right]$$

$$\boxed{\int_0^T e^{-x^2}dx \approx \frac{\sqrt{\pi}}{2} - \frac{e^{-T^2}}{2T}}$$