1)

~~2~~ a)

$V = \int_c^d 2\pi (\text{shell height})(\text{shell radius})\, dy$

$\int_0^1 2\pi y \,[1-(y-y^3)]\, dy$

$= 2\pi \int_0^1 (y - y^2 + y^4)\, dy = 2\pi \left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{y^5}{5}\right]_0^1$

$= 2\pi \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{5}\right) = \frac{2\pi}{30}(15-10+6) = \frac{11\pi}{15}$

b) washer method

$V = \int_c^d \pi [R^2(y) - r^2(y)]\, dy$

$= \int_0^1 \pi [1^2 - (y-y^3)^2]\, dy = \pi \int_0^1 (1 - y^2 - y^6 + 2y^4)\, dy$

$= \pi \left[y - \frac{y^3}{3} - \frac{y^7}{7} + \frac{2y^5}{5}\right]_0^1 = \pi\left(1 - \frac{1}{3} - \frac{1}{7} + \frac{2}{5}\right)$

$= \frac{97\pi}{105}$

c) use washer method

$V = \int_c^d \pi [R^2(y) - r^2(y)]\, dy$

$= \int_0^1 \pi [[1-(y-y^3)]]^2 - 0]\, dy$

$= \pi \int_0^1 [1 - 2(y-y^3) + (y-y^3)^2]\, dy$

$\rightarrow = \pi \int_0^1 (1 + y^2 + y^6 - 2y + 2y^3 - 2y^4)\, dy$

$= \pi \left[y + \frac{y^3}{3} + \frac{y^7}{7} - y^2 + \frac{y^4}{2} - \frac{2y^5}{5}\right]_0^1$

$= \frac{121\pi}{210}$

d) $V = \int_c^d 2\pi (h)(r)\, dy = \int_0^1 2\pi (1-y)[1-(y-y^3)]\, dy$

$= 2\pi \int_0^1 (1 - 2y + y^2 + y^3)\, dy = 2\pi \left[y - y^2 + \frac{y^3}{3} + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$

$= \frac{2\pi}{60}(20+15-12) = \frac{23\pi}{30}$

b) length of a curve

$$L = \int_a^b \sqrt{1+(f'(x))^2}\ dx \qquad f'(x) = \frac{dy}{dx}$$

$$L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

c) Surface Area

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$S = \int_c^d 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$f(x) = x \tan^{-1}(x^2)$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad ; \quad |x| \leq 1$$

$$(a^2)^4 = a^8$$

$$\tan^{-1}(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1}$$

$$x \tan^{-1}(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x \ x^{4n+2}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{2n+1}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{4(n+1)+3}}{2(n+1)+1}}{\frac{(-1)^n x^{4n+3}}{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^{4n+7}}{x^{4n+3}} \right| \left| \frac{2n+1}{2(n+1)+1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2n+1}{2(n+1)+1} \right| x^4$$

need $|x^4| < 1$

$$|x|^4 < 1$$

$$|x| < 1$$

$$\boxed{-1 < x < 1}$$

$$f(x) = \frac{x}{\frac{1}{x} + 1} \qquad \frac{1}{1 \mp x} = \sum_{n=0}^{\infty} x^n$$

$$= x \frac{1}{1-(-\frac{1}{x})} \Rightarrow$$

$$\frac{1}{1-(-\frac{1}{x})} = \sum_{n=0}^{\infty} \underline{(-1)^n (\frac{1}{x})^n}$$

$$\frac{x}{1+\frac{1}{x}} = \sum_{n=0}^{\infty} \underline{(-1)^n x^{-n} x} = \sum_{n=0}^{\infty} (-1)^n x^{1-n}$$

Radius of convergence

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{(1-(n+1))}}{(-1)^n x^{1-n}} \right| = \lim_{n\to\infty} \left| \frac{x^{-n}}{x^1 x^{-n}} \right|$$

$$\left| \frac{1}{x} \right| < 1$$

$x \neq 0$ - not defined for $x = 0$

$f(0) = $ undefined

$\underline{\sum (-1)^n (\frac{1}{x})^n}$ - not defined $x=0$

Radius of Convergence $= 0$

re-constr

$$f(x) = \frac{x^2}{1+x} = x^2 \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= \sum_{n=0}^{\infty} (-1) x^{n+2} \qquad |x|<1$$

b) True

$$\int_{\frac{2}{\sqrt{3}}}^{2} t\sec^{-1}(t)\,dt$$

1) $u = \sec^{-1} t$ $\qquad v = t$

$du = \dfrac{1}{t\sqrt{t^2-1}}\,dt \qquad dv = \dfrac{t^2}{2}$

$$\int_{\frac{2}{\sqrt{3}}}^{2} t\sec^{-1}(t)\,dt = \left[\frac{t^2}{2}\sec^{-1}(t)\right]_{\frac{2}{\sqrt{3}}}^{2} - \int_{\frac{2}{\sqrt{3}}}^{2} \frac{t^2}{2}\left[\frac{1}{t\sqrt{t^2-1}}\right]dt$$

$$= \left(2\cdot\frac{\pi}{3} - \frac{2}{3}\cdot\frac{\pi}{6}\right) - \int_{2/\sqrt{3}}^{2} \frac{t}{2\sqrt{t^2-1}}\,dt$$

u-sub

$$= \frac{5\pi}{9} - \left[\frac{1}{2}\sqrt{t^2-1}\right]_{2/\sqrt{3}}^{2} = \frac{5\pi - 3\sqrt{3}}{9}$$

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2) $\displaystyle\int \frac{6}{(9t^2+1)^2}\,dt \Rightarrow t = \frac{1}{3}\tan(\theta) \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dt = \frac{1}{3}\sec^2\theta\,d\theta$

$9t^2+1 = \sec^2\theta$

$$\int \frac{6}{(9t^2+1)^2}\,dt = \int \frac{6\left(\frac{1}{3}\sec^2\theta\right)}{\sec^4(\theta)}\,d\theta$$

$$= 2\int \cos^2\theta\,d\theta = \theta + \sin\theta\cos\theta + C$$

$$= \theta + \tan\theta\cos^2\theta + C$$

$$= \tan^{-1}(3t) + \frac{3t}{(9t^2+1)} + C$$

$$\int \frac{-2t^2+4t+6}{t^3+2t^2+t+2}\,dt$$

$$\cancel{\frac{-2t^2+4t+6}{t^3+2t^2+t+2} = \frac{A}{(t^2+1)} + \frac{B}{t+2} = A(t+2) + B(t^2+1)}$$

$$\cancel{-2t^2+4t+6 = At + 2A + Bt^2 + B}$$

$$\cancel{t^2:\quad -2 = B}$$

$$\cancel{t:\quad 4 = A}$$

$$\cancel{t^0:\quad 6 = 2A+B = 2\cdot 4 - 2 = 6}$$

Answer is correct

Method is not

$$\Rightarrow \int \frac{4}{t^2+1}\,dt + \int \frac{-2}{t+2}\,dt$$

$$= 4\tan^{-1}(t) + C - 2\ln(t+2)$$

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$$\int e^\theta \sin(\theta)\,d\theta \qquad \text{let} \quad dv = e^\theta\,d\theta,\quad v = e^\theta,\quad u = \sin\theta,\quad du = \cos\theta\,d\theta$$

$$I = e^\theta \sin\theta - \int e^\theta \cos\theta\,d\theta$$

$$I = e^\theta \sin\theta - \left[e^\theta \cos\theta - \int e^\theta(-\sin\theta)\,d\theta\right] \qquad \text{let} \quad dv = e^\theta\,d\theta,\quad v = e^\theta,\quad u = \cos\theta,\quad du = -\sin\theta\,d\theta$$

$$I = e^\theta \sin\theta - e^\theta\cos\theta - \int e^\theta \sin\theta\,d\theta$$

$$I = e^\theta(\sin\theta - \cos\theta) - I$$

$$2I = e^\theta(\sin\theta - \cos\theta)$$

$$I = \frac{e^\theta}{2}(\sin\theta - \cos\theta)$$

$$\int \frac{5-x}{x^2-1}\,dx$$

$$\frac{5-x}{x^2-1} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$$

$$5-x = Ax - A + Bx + B$$

$$5 = B - A \quad \Rightarrow \quad B = 5 + A$$
$$-1 = A + B$$

$$-1 = A + 5 + A$$
$$-1 = 2A + 5$$
$$-6 = 2A$$
$$\boxed{-3 = A}$$

$$5 = B + 3$$
$$2 = B$$

$$\int \frac{5-x}{x^2-1}\,dx = \int \frac{-3}{x+1} + \frac{2}{x-1}\,dx$$

$$= -3\ln(x+1) + 2\ln(x-1) + C$$

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$$\int \frac{(1-r^2)^{5/2}}{r^8}\,dr \qquad \text{let } r = \sin\theta \quad -\pi/2 < \theta < \pi/2$$

$$\int \frac{\cos^5\theta \cos\theta}{\sin^8\theta}\,d\theta = \int \cot^6\theta \csc^2\theta\,d\theta$$

$$= -\frac{\cot^7\theta}{7} + C = -\frac{1}{7}\left[\frac{\sqrt{1-r^2}}{r}\right]^7 + C$$

3(a)

7) $f(x) = \ln(1+\sqrt{x})$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad |x|<1$$

$$f(x) = \ln(1+\sqrt{x}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x^{\frac{n}{2}})}{n}$$

Radius of convergen

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^n (x^{\frac{(n+1)}{2}})}{n+1}}{\frac{(-1)^{n-1} x^{n/2}}{n}} \right| = \lim_{n\to\infty} \left| \frac{x^{n/2} \, x^{1/2}}{x^{n/2}} \right| \left| \frac{n}{n+1} \right|$$

$$\cancel{0} \leq |x^{\frac{1}{2}}| < 1$$

$$0 < x^{\frac{1}{2}} < 1$$

$$0 < x < 1 \quad \text{-check endpts}$$

$$0 \rightarrow f(0) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(0)}{n} = 0$$

-converges absolutily

$$1 \rightarrow f(1) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n}$$

∴ converges conditionaly -Alternating series

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Absolutely converges $0 \leq x < 1$

converges conditionally $x=1$

i) a) $x = r\cos\theta$ $\Rightarrow r^2 = x^2+y^2$

$y = r\sin\theta$ $\tan\theta = y/x$

$r = \sqrt{x^2+y^2}$

$r = \theta \Rightarrow$

$\theta = \tan^{-1}(y/x)$

$\sqrt{x^2+y^2} = \tan^{-1}(y/x)$

ii) when we cross x-axis $y=0$

$y = r\sin\theta = \theta \cdot \sin\theta \Rightarrow \left\{ \begin{matrix} \theta = 0 \\ \sin\theta = 0 \end{matrix} \right\}$

$\theta = 2n\pi \quad n = 0, 1, 2, \ldots$

-positive x-axis

$P_1(0,0)$, $P_2 = (2\pi, 0)$, $P_3 = (4\pi, 0)$

c) Area $= \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \theta^2 \, d\theta$

$= \frac{1}{2} \frac{\theta^3}{3} \Big|_0^{2\pi} \quad \frac{4\pi^3}{3}$

d) length $= \int_0^{4\pi} \sqrt{r^2 + (r')^2} \, d\theta = \int_0^{4\pi} \sqrt{1+\theta^2} \, d\theta$

$r = \theta \Rightarrow r' = 1$