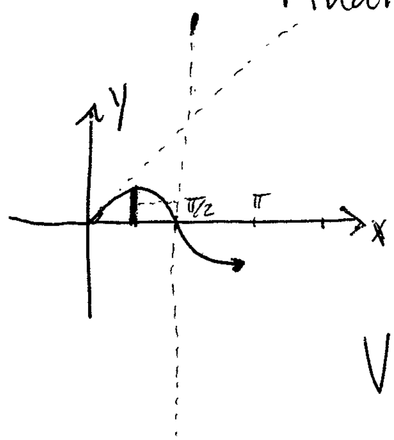
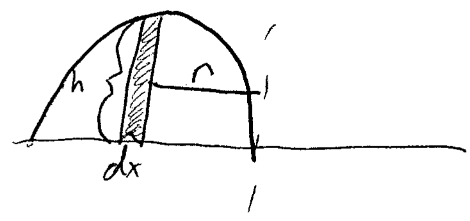
Final Exam Solutions

1. a)



Shells:



$$V = 2\pi\int_0^{\pi/2} r h\,dx = \boxed{2\pi\int_0^{\pi/2}\left(\frac{\pi}{2}-x\right)x\cos(x)\,dx}$$

(b) Evaluate: $2\pi\int_0^{\pi/2} \frac{\pi}{2}x\cos(x) - x^2\cos(x)\,dx =$

$$\pi^2\int_0^{\pi/2} x\cos(x)\,dx - 2\pi\int_0^{\pi/2} x^2\cos(x)\,dx$$

$$\int_0^{\pi/2} x\cos(x)\,dx = \Big[x\sin(x) - \int\sin(x)\,dx\Big]_0^{\pi/2} = \Big[x\sin(x) + \cos(x)\Big]_0^{\pi/2} = \frac{\pi}{2} - 1$$

$$\int_0^{\pi/2} x^2\cos(x)\,dx = \Big[x^2\sin(x) - 2\int x\sin(x)\,dx\Big]_0^{\pi/2} = \Big[x^2\sin(x) - 2\big(-x\cos(x) + \int\cos(x)\,dx\big)\Big]_0^{\pi/2}$$

$$= \Big[x^2\sin(x) + 2x\cos(x) - 2\sin(x)\Big]_0^{\pi/2} = \frac{\pi^2}{4} - 2$$

$$\pi^2\int_0^{\pi/2} x\cos(x)\,dx - 2\pi\int_0^{\pi/2} x^2\cos(x)\,dx = \pi^2\left(\frac{\pi}{2}-1\right) - 2\pi\left(\frac{\pi^2}{4}-2\right) = \frac{\pi^3}{2} - \pi^2 - \frac{\pi^3}{2} + 4\pi$$

$$= \boxed{4\pi - \pi^2}$$

1.b) $x(t) = \frac{1}{V(t)} \int V(t) Q(t)\,dt$

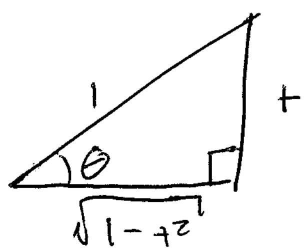
$P(t) = \frac{2}{t}$

$Q(t) = \frac{1}{1-t^2}$

$V(t) = e^{\int P(t)\,dt} = e^{2\ln(t)} = e^{\ln t^2} = t^2$

$x(t) = \frac{1}{t^2} \int \frac{t^2}{1-t^2}\,dt$

(d) Evaluate: $\frac{1}{t^2} \int \frac{t^2}{1-t^2}\,dt = \frac{1}{t^2} \int \tan^2\theta \cos\theta\,d\theta$



$t = \sin\theta \qquad \frac{t}{\sqrt{1-t^2}} = \tan\theta$

$dt = \cos\theta\,d\theta \qquad \frac{t^2}{1-t^2} = \tan^2\theta$

$\frac{1}{t^2} \int \frac{\sin^2\theta}{\cos^2\theta} \cos\theta\,d\theta = \frac{1}{t^2} \int \frac{\sin^2\theta}{\cos\theta}\,d\theta = \frac{1}{t^2} \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta =$

$= \frac{1}{t^2} \int \sec\theta - \cos\theta\,d\theta = \frac{1}{t^2}\left[\ln|\sec\theta + \tan\theta| - \sin\theta + c\right]$

$$= \boxed{\frac{1}{t^2}\left[\ln\left|\frac{1}{\sqrt{1-t^2}} + \frac{t}{\sqrt{1-t^2}}\right| - t + c\right]}$$

1. c) $L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \boxed{\int_0^{\pi} \sqrt{(-\sin(t))^2 + (1+\cos(t))^2}\, dt}$

$\frac{dx}{dt} = -\sin(t)$

$\frac{dy}{dt} = 1 + \cos(t)$

(d) Evaluate: $\int_0^{\pi} \sqrt{\sin^2(t) + 1 + 2\cos(t) + \cos^2(t)}\, dt$

$= \int_0^{\pi} \sqrt{2 + 2\cos(t)}\, dt = \int_0^{\pi} \sqrt{4\cos^2(\frac{t}{2})}\, dt = \int_0^{\pi} 2|\cos(\frac{t}{2})|\, dt = \int_0^{\pi} 2\cos(\frac{t}{2})\, dt$

[using $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$: multiply by 4 and let $\theta = \frac{t}{2}$ to get $4\cos^2(\frac{t}{2}) = 2 + 2\cos(t)$]

$= 2\left[2\sin(\frac{t}{2})\right]_0^{\pi} = 2[2 - 0] = \boxed{4}$

2. a) $f(x) = \arctan(x)$

$$f'(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$ [by plugging into $\sum_0^\infty (-y)^n = \frac{1}{1+y}$ with $y = x^2$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

$$\int \frac{dx}{1+x^2} = \arctan(x) = \int 1 - x^2 + x^4 - x^6 + \dots \, dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad \sum_0^\infty \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

b) $\sum_0^\infty \frac{(-1)^n x^{2n+1}}{2n+1}$ ACT: $\sum_0^\infty \left| \frac{(-1)^n x^{2n+1}}{2n+1} \right| = \sum_0^\infty \frac{|x|^{2n+1}}{2n+1}$

RT: $\underbrace{\frac{|x|^{2n+3}}{2n+3}}_{a_{n+1}} \cdot \underbrace{\frac{2n+1}{|x|^{2n+1}}}_{1/a_n} = |x|^2 \frac{2n+1}{2n+3} \xrightarrow{\text{L'Hop}} |x|^2$

require $|x|^2 < 1$ so $-1 < x < 1$

endpoints: @ $x = -1$: $\sum_0^\infty \frac{(-1)^n (-1)^{2n+1}}{2n+1} = \sum_0^\infty \frac{(-1)^n (-1)^{2n} (-1)}{2n+1} \overset{!!}{=} \sum_0^\infty \frac{(-1)^n}{2n+1}$

AST: 1. $u_n = \frac{1}{2n+1} > 0$ ✓

2. $u_{n+1} \le u_n$: $\frac{1}{2n+3} < \frac{1}{2n+1} \Leftrightarrow 2n+1 < 2n+3$ ✓

3. $u_n = \frac{1}{2n+1} \to 0$ ✓

Converges at $x = -1$

same argument at $x = 1$: $\sum \frac{(-1)^n}{2n+1}$ converges

$\boxed{\text{IOC: } -1 \le x \le 1}$

2.c) $P_3(x) = x - \frac{x^3}{3}$

$\arctan(.1) = f(.1) \approx P_3(.1) = \boxed{.1 - \frac{(.1)^3}{3}}$

2.d) The series converges for $|x| \le 1$, so

$\arctan(.1) = .1 - \frac{(.1)^3}{3} + \frac{(.1)^5}{5} - \frac{(.1)^7}{7} + \dots$

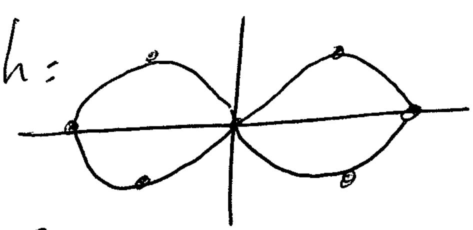
This series alternates, so by the ASET,

$\boxed{|\text{error}| \le \left|\frac{(.1)^5}{5}\right| = \frac{(.1)^5}{5}}$

3.a) $\left.\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}\right\}$ so $r = 1 + \cos(2\theta)$

becomes $\boxed{\sqrt{x^2+y^2} = 1 + \cos\left(2\tan^{-1}\left(\frac{y}{x}\right)\right)}$

b) $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$ Graph:



$A = \frac{1}{2}\int_0^{2\pi} (1+\cos(2\theta))^2\,d\theta = \frac{1}{2}\int_0^{2\pi} 1 + 2\cos(2\theta) + \cos^2(2\theta)\,d\theta$

$= \frac{1}{2}\int_0^{2\pi} 1 + 2\cos(2\theta) + \frac{1}{2}(1 + \cos(4\theta))\,d\theta$

$= \frac{1}{2}\left[\theta + \sin(2\theta) + \frac{\theta}{2} + \frac{\sin(4\theta)}{8}\right]_0^{2\pi}$

$= \frac{1}{2}\left[2\pi + \pi\right] = \boxed{\frac{3\pi}{2}}$

4. (a) $\sum_{3}^{\infty} \frac{n+1}{\sqrt{n^3-2}}$ LCT w/ $\frac{1}{n^{1/2}}$

$$\frac{\frac{n+1}{\sqrt{n^3-2}}}{\frac{1}{n^{1/2}}} = \frac{n^{3/2}+n^{1/2}}{\sqrt{n^3-2}} = \frac{1+\frac{1}{n}}{\frac{1}{n^{3/2}}\sqrt{n^3-2}} = \frac{1+\frac{1}{n}}{\sqrt{1-\frac{2}{n^3}}}$$

$\lim_{n\to\infty} \frac{1+\frac{1}{n}}{\sqrt{1-\frac{2}{n^3}}} = 1$. Since $\sum_{1}^{\infty} \frac{1}{n^{1/2}}$ diverges,

$\boxed{\sum_{3}^{\infty} \frac{n+1}{\sqrt{n^3-2}} \text{ diverges}}$ too

b) $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \left[2x^{1/2}\right]_a^1 = \lim_{a\to 0^+} \left[2-2a^{1/2}\right]$
$= 2$

$\boxed{\text{Converges}}$

5. $\sum_{0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n$ is geometric with $|r| = \left|\frac{2}{3}\right| < 1$

so it converges. It converges to $\frac{1/3}{1-2/3} = \frac{1/3}{1/3} = \boxed{1}$

The length of the original interval was 1, and we took out something of length 1 so what is left has length 0, or is not measureable. However the numbers 0, 1/3, 2/3, 1, 1/9, 2/9, 7/9, 8/9, and many others are never taken out. It turns out that there are quite a lot left.