Exam 2 Fall 2007 Solutions

① a) $\int_1^e x \ln x \, dx = \frac{x^2}{2} \ln x \Big|_1^e - \int_1^e \frac{x}{2} \, dx$

$x = dv \quad u = \ln x$

$v = \frac{x^2}{2} \quad du = \frac{1}{x}$

$= \frac{e^2}{2} - \frac{x^2}{4}\Big|_1^e = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}$

$= \frac{e^2}{4} + \frac{1}{4}$

b) $\int_0^{1/2} \frac{6}{\sqrt{1-4x^2}} \, dx = \lim_{b \to 1/2^-} \int_0^b \frac{6}{\sqrt{1-4x^2}} \, dx$

let $x = \frac{1}{2}\sin\theta$

$dx = \frac{1}{2}\cos\theta \, d\theta$

$= \lim_{b \to \pi/2} \int_0^b \frac{3\cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}}$

$= \int_0^{\pi/2} 3 \, d\theta = 3\theta \Big|_0^{\pi/2} = \frac{3\pi}{2}$

c) $\int_{-1}^1 \frac{1}{t^3} \, dt = \int_{-1}^0 \frac{1}{t^3} \, dt + \int_0^1 \frac{1}{t^3} \, dt$

$\int_{-1}^0 \frac{1}{t^3} \, dt = \lim_{b \to 0^-} \int_{-1}^b \frac{1}{t^3} \, dt = \lim_{b \to 0^-} \frac{-1}{2t^2} \Big|_{-1}^b$

$= \lim_{b \to 0^-} \frac{-1}{2b^2} + \frac{1}{2} = \infty$ diverges

Thus $\int_{-1}^1 \frac{1}{t^3} \, dt$ diverges

(2) a) $a_n = 1 + (-1)^n = \begin{cases} 0 & \text{for } n \text{ odd} \\ 2 & \text{for } n \text{ even} \end{cases}$

$\Rightarrow \text{Limit}_{n\to\infty}$ DNE $\Rightarrow$ Diverges

b) $$b_n = \frac{1}{\sqrt{n^2+n} - \sqrt{n^2+1}} \cdot \frac{\sqrt{n^2+n} + \sqrt{n^2+1}}{\sqrt{n^2+n} + \sqrt{n^2+1}}$$

$$= \frac{\sqrt{n^2+n} + \sqrt{n^2+1}}{(\cancel{n^2}+n) - (\cancel{n^2}+1)} = \frac{\sqrt{n^2+n} + \sqrt{n^2+1}}{n-1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \frac{\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{n^2}}}{1 - \frac{1}{n}}$$

$$\lim_{n\to\infty} b_n = \frac{\sqrt{1}+\sqrt{1}}{1} = 2$$ converges to 2

c) $c_n = n \ln\left(1 + \frac{2}{n}\right)$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}} \overset{\text{L'hopital}}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{2}{n}} \cdot \frac{\cancel{-}2}{n^2}}{\cancel{-}1/n^2}$$

$$= \lim_{n\to\infty} \frac{2}{1+\frac{2}{n}} = 2$$

3 a) $\{a_n\} = \left\{ \frac{3}{n(n+1)} \right\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{n(n+1)} = 0$$

$\Rightarrow$ sequence converges to 0

b) $S_n = ?$

$$\frac{3}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$A(n+1) + Bn = 3$

$\underbrace{(A+B)}_{0}n + \underbrace{A}_{3} = 3$

$A = -B$

$$= \frac{3}{n} - \frac{3}{n+1}$$

$A = 3$

$B = -3$

$$S_n = \sum_{n=1}^{n} a_n = \sum_{n=1}^{n} \frac{3}{n} - \frac{3}{n+1}$$

$$S_n = \underbrace{\left(\frac{3}{1} - \cancel{\frac{3}{2}}\right)}_{n=1} + \underbrace{\left(\cancel{\frac{3}{2}} - \cancel{\frac{3}{3}}\right)}_{n=2} \cancel{+} \cancel{\cdots} \cancel{+} \underbrace{\left(\cancel{\frac{3}{n-1}} - \cancel{\frac{3}{n}}\right)}_{n-1} + \left(\cancel{\frac{3}{n}} - \frac{3}{n+1}\right)$$

$$\boxed{S_n = 3 - \frac{3}{n+1}}$$

c) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 3 - \frac{3}{n+1} = 3$

$\Rightarrow$ sequence $\{S_n\}$ converges to 3

d) $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = 3$

$\Rightarrow$ series converges to 3

4 a) $0 \le a_n \le 2a_n + 3b_n$

$\sum_{n=1}^{\infty} (2a_n + 3b_n)$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges by DCT

$0 \le b_n \le 2a_n + 3b_n$

$\sum_{n=1}^{\infty} (2a_n + 3b_n)$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ converges by DCT

TRUE

b) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\left(\begin{array}{c}\text{p-series} \\ p=1\end{array}\right)$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\left(\begin{array}{c}\text{p-series} \\ p=2>1\end{array}\right)$

$\sum_{n=1}^{\infty} (2a_n + 3b_n) = \sum_{n=1}^{\infty} \left(\frac{2}{n} + \frac{3}{n^2}\right) = \sum_{n=1}^{\infty} \left(\frac{2n+3}{n^2}\right)$ diverges

$\frac{2}{n} \le \frac{2}{n} + \frac{3}{n^2}$ $\quad$ $\sum \frac{2}{n}$ diverges $\Rightarrow \sum \frac{2n+3}{n^2}$ diverges by DCT

FALSE

(5) a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n^3))} = \sum_{n=2}^{\infty} \frac{1}{n \cdot 3\ln(n)}$

Integral Test

$$\frac{1}{3}\int_2^{\infty} \frac{1}{x\ln x}\,dx = \frac{1}{3}\int_{\ln 2}^{\infty} \frac{du}{u} = \frac{1}{3}\ln u \Big|_{\ln 2}^{\infty}$$

let $u = \ln x$
$du = \frac{1}{x}dx$

$$= \lim_{b\to\infty} \frac{1}{3}\ln b - \frac{1}{3}\ln(\ln 2) = \infty$$

Diverges

b) $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n}$ $\cos(n\pi) = (-1)^n$

$= \sum_{n=0}^{\infty} \left(\frac{-1}{5}\right)^n$ geometric: $r = -\frac{1}{5}$ $|r| < 1$
$a = 1$

$|r| < 1 \Rightarrow$ Converges

c) $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

$\sum_{n=1}^{\infty} \frac{1}{1+2+3+\ldots+n} = \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \leq \sum_{n=1}^{\infty} \frac{2}{n^2} \leftarrow$ converges p-series $p = 2 > 1$

Converges by DCT