Exam #3 Fall 2007 Solutions

(1) a) $1+2+3+\dots+n = \frac{n}{2}(n+1)$

$$\sum_{n=1}^{\infty} \frac{1}{1+2+3+\dots+n} = \sum_{n=1}^{\infty} \frac{2}{n(n+1)}$$

$$\frac{2}{n(n+1)} \leq \frac{2}{n^2}$$

$\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{2}{n(n+1)}$ $\boxed{\text{converges}}$ by DCT

b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ $\qquad u_n = \frac{1}{n}$

1. $u_n \geq 0$ ✓
2. $\frac{1}{n} \geq \frac{1}{n+1} \Rightarrow u_n \geq u_{n+1}$ ✓
3. $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$\boxed{\text{Converges}}$ by AST

(Note: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow$ conditional convergence)

c) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$

$$\sin x - x + \frac{x^3}{6} = \left(\cancel{x} - \cancel{\frac{x^3}{6}} + \frac{x^5}{5!} \dots\right) - \cancel{x} + \cancel{\frac{x^3}{6}}$$

$$= \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \dots$$

Thus $\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0} \frac{1}{x^5}\left(\frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \dots\right)$

$$= \lim_{x\to 0}\left(\frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} \dots\right)$$

$$= \boxed{\frac{1}{5!} = \frac{1}{120}}$$

② $\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$

a) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{x^n} \right|$

$= \lim_{n\to\infty} |x| \frac{\cancel{n}^{\,1}}{\cancel{n}+1} \cdot \frac{\ln n}{\ln(n+1)}$

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \overset{L.H.}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

Thus $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| < 1$

$\therefore$ $\boxed{\text{Radius of convergence} = 1}$

b) Test endpoints:

$x = 1$: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

integral test: $\int_2^{\infty} \frac{1}{x \ln x}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u}\,du = \ln u \Big|_{\ln 2}^{\infty} = \infty$

let $u = \ln x$, $du = \frac{1}{x}\,dx$ diverges

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges

$x = -1$: $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$ diverges absolutely (see above)

absolute convergence for $\boxed{-1 < x < 1}$

c) Test $x = -1$ for conditional convergence

AST: $u_n = \frac{1}{n \ln n}$

1. $u_n = \frac{1}{n \ln n} \geq 0$ for $n \geq 2$ ✓
2. $\frac{1}{n \ln n} \geq \frac{1}{(n+1)\ln(n+1)} \Rightarrow u_n \geq u_{n+1}$ ✓
3. $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$ ✓

conditional convergence for $\boxed{x = -1}$

d) diverges for $x \geq 1$ and $x < -1$

$(-\infty, -1) \cup [1, \infty)$

③ $-1 < x < 1$

a) $a = 1$

$r = x \qquad |r| < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

b) differentiate

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$\frac{+1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$$

c) multiply by x

$$\frac{x}{(1-x)^2} = x \sum_{n=1}^{\infty} n x^{n-1} = x \sum_{n=0}^{\infty} (n+1) x^n$$

$$= \sum_{n=1}^{\infty} n x^n = \sum_{n=0}^{\infty} (n+1) x^{n+1}$$

d) $$\sum_{n=1}^{\infty} \frac{n}{3^n} = \sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^n = \frac{1/3}{(1-1/3)^2} = \frac{1}{3}\cdot\left(\frac{3}{2}\right)^2 = \boxed{\frac{3}{4}}$$

$x = 1/3$ in part c

4) $f(t) = \cos(t^2)$

a) $\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}$

$$\cos(t^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(t^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n}}{(2n)!}$$

$$= 1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \dots$$

b) $\cos(t^2) \approx 1 - \frac{t^4}{2!} = P_4(t)$

$$\int_0^1 \cos(t^2)\,dt \approx \int_0^1 \left(1 - \frac{t^4}{2!}\right) dt = t - \frac{t^5}{5 \cdot 2!} \Big|_0^1$$

$$= 1 - \frac{1}{5 \cdot 2} = 1 - \frac{1}{10} = \boxed{\frac{9}{10}}$$

c) $|\text{alternating series error}| \leq \left| \begin{matrix}\text{first term} \\ \text{not kept}\end{matrix} \right| = \int_0^1 \frac{t^8}{4!}\,dt$

$$|\text{error}| \leq \frac{t^9}{9 \cdot 4!} \Big|_0^1 = \boxed{\frac{1}{9 \cdot 4!}}$$

d) first term not kept $= \int_0^1 \frac{t^8}{4!}\,dt$ which is positive

$\Rightarrow$ estimate is an underestimate of exact value

(5) $\dfrac{x^2}{c-4} + \dfrac{y^2}{10-c} = 1$

a) $c < 4$

$\Rightarrow$ $c-4 < 0$, $10-c > 0$ $\Rightarrow$ $\dfrac{-x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $\boxed{\text{hyperbola}}$

$a^2 \neq b^2$

b) $4 < c < 10$

$\Rightarrow$ $c-4 > 0$, $10-c > 0$ $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

[Note: if $c-4 = 10-c \Rightarrow 2c = 14 \Rightarrow c = 7$

at $c=7$ $a^2 = b^2 \Rightarrow$ circle]

if $c \neq 7$ $\boxed{\text{ellipse}}$

c) $10 < c$

$\Rightarrow$ $c-4 > 0$, $10-c < 0$ $\Rightarrow$ $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ $\boxed{\text{hyperbola}}$