1.a

$$\lim \frac{\frac{|x+3|^{n+1}}{\sqrt{n+1}}}{\frac{|x+3|^n}{\sqrt{n}}} = \lim |x+3| \frac{\sqrt{n}}{\sqrt{n+1}} \overset{DP}{=} \lim |x+3| \frac{\sqrt{n}}{\sqrt{n}}$$

$$= |x+3|$$

Converges absolute for $|x+3|<1$.

$|x+3| = \pm 1$,

Absolute? $\sum \frac{1}{\sqrt{n}}$ diverges p-series test with $p = \frac{1}{2} \leq 1$.

Conditional, at $x+3=1$

$\sum \frac{1}{\sqrt{n}}$ diverges

at $x+3=-1$

$\sum \frac{(-1)^n}{\sqrt{n}}$, $\frac{1}{\sqrt{n}} \searrow$, $\lim \frac{1}{\sqrt{n}} = 0$

$\frac{1}{\sqrt{n}} > 0$

AST $\Rightarrow \sum \frac{(-1)^n}{\sqrt{n}}$ converges.

So converges absolutely for $|x+3|<1$
converges conditionally for $x+3=-1$
diverges for all other x.

Q 1.b. $\cos n\pi = (-1)^n$

$$\sum \frac{\cos n\pi}{3^n} x^n = \sum (-1)^n \left(\frac{x}{3}\right)^n \quad \text{geometric}$$

Converges absolutely $|x|<3$
Diverges for $|x| \geq 3$.

1C. $$\lim \frac{\frac{|2x-8|^{n+1}}{(n+1)!}}{\frac{|(2x-8)|^n}{n!}} = \lim_{n\to\infty} \frac{|2x-8|}{n+1} = 0$$

converges absolutely for all x.

2. a) $$\int_0^1 \sin t^3 \, dt = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n (t^3)^{2n+1}}{(2n+1)!} \, dt$$

$$= \sum_{n=0}^{\infty} \left(\int_0^1 t^{6n+3} \, dt \right) \cdot \frac{(-1)^n}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(6n+4)}$$

$$= \frac{1}{4} - \frac{1}{10 \cdot 3!} + \frac{1}{16 \cdot 120} - \underbrace{\boxed{\frac{1}{22 \cdot 7!}}}_{\leq \frac{1}{10^4}} + \cdots$$

Need 3 non-zero terms. by Alternating series estimate theorem.

b) $$\lim_{x\to 0} \frac{\cos x^3 - 1 + \frac{x^6}{2}}{2x^{12}} = \lim_{x\to 0} \frac{1 - \frac{x^6}{2} + \frac{x^{12}}{4!} + \cdots - 1 + \frac{x^6}{2}}{2x^{12}}$$

$$= \lim_{x\to 0} \frac{x^{12}}{2x^{12} \cdot 4!} = \boxed{\frac{1}{48}}$$

3a) $f(x) = \frac{1}{1+x}$, $f'(x) = \frac{-1}{(1+x)^2}$, $f''(x) = \frac{1 \cdot 2}{(1+x)^3}$, $f'''(x) = \frac{-3!}{(1+x)^4}$,

$\cdots$ $f^{(n)}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$ $\cdots$ $f^{(n)}(0) = (-1)^n n!$

$$MS = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} x^n$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n x^n}$$

3b) $e^x = \sum_0^\infty \frac{x^n}{n!}$, $e^{x^2} = \sum_0^\infty \frac{x^{2n}}{n!}$

$$\int_0^1 e^{x^2} \cong \int_0^1 \left(1+\frac{x^2}{1}+\frac{x^4}{2}\right)dx = \boxed{1+\frac{1}{3}+\frac{1}{10}}$$

3c) $e^x = 1 + e^c x$ $\quad 0 \le c \le x$

$$e^{t^2} = 1 + e^c t^2 \qquad 0 \le c \le t^2 \le 1$$

The error is

$$\int_0^1 e^c t^2 dt \le \int_0^1 e^1 t^2 dt = \boxed{\frac{e}{3}}$$

Or some might try:

$$f(x) = e^{x^2}, \quad f(0) = 1, \quad f'(x) = 2xe^{x^2}.$$

$$R_0(x) = f'(c)x = 2ce^{c^2}x$$

$$0 \le R_0(x) \le 2e^1 x \quad (0 \le c \le 1).$$

The error

$$\le \int_0^1 R_0(x)\,dx \le \int_0^1 2ex\,dx = \boxed{e}$$

Or $f(x) = \int_0^x e^{t^2} dt$, $f(0) = 0$, $\boxed{\text{first term}}$

$$R_0(\cancel{x}) = f'(c)\cancel{x} = e^{c^2}\cdot 1 \le \boxed{e}$$

4 <u>Direct</u> $f(x) = (1-3x)^{1/2}$, $f(0) = 1$

$f'(x) = \dfrac{-3}{2(1-3x)^{1/2}}$, $f'(0) = \dfrac{-3}{2}$

$f''(x) = \dfrac{-9}{4(1-3x)^{3/2}}$, $f''(0) = -\dfrac{9}{4}$.

$f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2}x^2 + \cdots$

$= \boxed{1 - \dfrac{3}{2}x - \dfrac{9}{8}x^2} + \cdots$

Or $(1+x)^{1/2} = 1 + \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \cdots$

$= 1 + \dfrac{\frac{1}{2}}{1}x + \dfrac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^2 + \cdots$

$= 1 + \dfrac{x}{2} - \dfrac{1}{8}x^2 + \cdots$

$(1-3x) = 1 + \frac{1}{2}(-3x) - \frac{1}{8}(-3x)^2 + \cdots$

$= \boxed{1 - \dfrac{3}{2}x - \dfrac{9}{8}x^2} + \cdots$

5 (a) i) Hyperbola $(q-5<0,\ 8-q>0)$
ii) ellipse $(q-5>0,\ 8-q>0)$
iii) Hyperbola $(q-5>0,\ 8-q<0)$

(c) $(\pm 4, 0)$ — Vertices $\qquad c = \sqrt{16+4} = \sqrt{20}$

$(\pm\sqrt{20}, 0)$ — Foci, $\qquad e = \dfrac{c}{a} = \boxed{\dfrac{\sqrt{20}}{4}}$

<u>Asymptotes</u> $y = \pm\frac{1}{2}\cancel{x}$

5b