ANSWER KEY, Test 2, Sp08 1360

1a) $$\int \frac{dt}{\sqrt{9-4t^2}} = \frac{1}{3}\int \frac{dt}{\sqrt{1-\frac{4t^2}{9}}}$$

$\frac{4t^2}{9} = \sin^2 x$

$\frac{2t}{3} = \sin x$

$t = \frac{3}{2}\sin x$

$dt = \frac{3}{2}\cos x\,dx$

$$\frac{3}{2}\cdot\frac{1}{3}\int \frac{\cancel{\cos x}\,dx}{\cancel{\sqrt{1-\sin^2 x}}}$$

$$= \boxed{\frac{1}{2}\arcsin\frac{2t}{3} + C}$$

1b) $$\int_e^\infty \frac{6}{x(\ln x)^2}\,dx = 6\int_{x=e}^{x=\infty} \frac{du}{u^2}$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= -\frac{6}{u}\Big|_{x=e}^{x=\infty} = \lim_{a\to\infty} -\frac{6}{\ln x}\Big|_e^a = \lim_{a\to\infty} \cancel{\frac{-6}{\ln a}}^{\,0} + \frac{6}{1} = \boxed{6}$$

1c) $$\int_{-2}^{0} \frac{1}{t}\,dt + \int_0^2 \frac{1}{t}\,dt$$

$$\int_0^2 \frac{1}{t}\,dt = \lim_{a\to 0^+} \int_a^2 \frac{1}{t}\,dt = \lim_{a\to 0^+} \ln 2 - \cancel{\ln a}^{\,-\infty}$$

$$= \ln 2 + \infty = \boxed{\infty} \text{ diverges}$$

2a) $\lim_{n\to\infty} \frac{n}{n+1} = 1$ hence the seq. oscillates btw. $-1, 1$ at ∞

therefore $\lim_{n\to\infty} \frac{(-1)^n n}{n+1}$ $\boxed{\text{diverges}}$

b) $$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+2n+1}-\sqrt{n^2+n+1}} \cdot \frac{\sqrt{n^2+2n+1}+\sqrt{n^2+n+1}}{\sqrt{n^2+2n+1}+\sqrt{n^2+n+1}}$$

$\boxed{\text{converges}}$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2+2n+1}+\sqrt{n^2+n+1}}{\cancel{n^2}+2n+\cancel{1}-\cancel{n^2}-n-\cancel{1}} = \lim_{n\to\infty} \sqrt{1+2/n+\frac{1}{n^2}} + \sqrt{1+1/n+\frac{1}{n^2}} = \boxed{2}$$

2c) $\lim_{n\to\infty} \ln\left(1+\frac{4}{n}\right)^n = \ln e^4$ (Table 8.1) $= \boxed{4}$ $\boxed{\text{converges}}$

3a) F Counter: $\sum \frac{(-1)^n}{\sqrt{n}}$ conv., $\sum \frac{(-1)^n}{\sqrt{n}}$ conv. $\sum 2 \frac{(-1)^n}{\sqrt{n}} \frac{(-1)^n}{\sqrt{n}} = \sum \frac{2}{n}$ div. H.S.

b) F $\sum \frac{1}{n}, \sum \frac{1}{\sqrt{n}}$ $\frac{1}{n} \leq \frac{1}{\sqrt{n}}$ $\forall n$ but $\sum \frac{1}{n}$ diverges

c) T

4a) $\sum_{n=2}^{\infty} (-1)^n \left(1+\frac{1}{n}\right)^n$ NTT $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$ $\boxed{\text{div. Absolute form}}$

fails abs convergence

AST: ① $\lim_{n\to\infty} a_n = e \neq 0$ (see above)

$\Rightarrow$ $\boxed{\text{diverges in all forms}}$ NTT

b) $\sum_{2}^{\infty} \frac{3}{n \ln n}$ IT: $3\int_{x=a}^{1} \frac{dx}{x \ln x} = \lim_{a\to\infty} 3\int_{x=1}^{x=a} \frac{du}{u}$ $\quad u = \ln x$, $du = \frac{1}{x}dx$

$= \lim_{a\to\infty} 3 \ln u \Big|_{x=1} = \lim_{a\to\infty} 3\left[\ln \ln a - \ln \ln 1\right] = 3(\infty + \infty) = \boxed{\infty}$

$\boxed{\text{diverges}}$ I.T.

c) $\sum_{1}^{\infty} \frac{\tan^{-1} n}{n^{3/2}}$ compare to $\sum \frac{1}{n^{3/2}}$ conv. PST $p = 3/2 > 1$

LCT $\lim_{n\to\infty} \frac{\frac{\tan^{-1} n}{\cancel{n^{3/2}}}}{\frac{1}{\cancel{n^{3/2}}}} = \lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$ $\Rightarrow$ $\boxed{\text{conv. LCT}}$

d) $\sum_{1}^{\infty} \frac{5^n}{n!}$ RT $\lim_{n\to\infty} \frac{\cancel{5^n}\, 5}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{5^n}} = \lim_{n\to\infty} \frac{5}{n+1} = 0 < 1$

$\Rightarrow$ $\boxed{\text{conv by Ratio Test}}$

5a) $\sum_{n=2}^{\infty} \frac{\tan^{-1} n}{n^2+1}$ compare to $\sum_{1}^{\infty} \frac{1}{n^2}$ conv. PST $p=2>1$

$$\lim_{n\to\infty} \frac{\frac{\tan^{-1} n}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \tan^{-1} n \; \frac{\cancel{n^2}}{\cancel{n^2}+1} \overset{\text{DOP}}{\nearrow}1 = \frac{\pi}{2}$$

$\Rightarrow$ $\boxed{\text{ABS convergence}}$

b) $\sum_{n=2}^{\infty} \left|\frac{2(-1)^n}{n \ln n}\right| = 2\sum \frac{1}{n \ln n} = \frac{2}{3}\sum \frac{3}{n \ln n}$

which diverges by 4b

AST: 1) $\lim_{n\to\infty} \frac{2}{n \ln n} = 0$ ✓

2) $\frac{2}{n \ln n} > 0$ since $n \geq 2$

3) WTS: $a_{n+1} < a_n$

① $1 > 0$, $n+1 > n$

② $\ln(n+1) > \ln n$

Thus $(n+1)\ln(n+1) > n \ln n$

$\Rightarrow \frac{2}{(n+1)\ln(n+1)} < \frac{2}{n \ln n}$ ✓ decr

Thus $\sum_{2}^{\infty} \frac{2(-1)^n}{n \ln n}$ $\boxed{\text{converges conditionally}}$

c) $\sum_{1}^{\infty} \frac{\cos n\pi}{n} = \sum_{1}^{\infty} \frac{(-1)^n}{n}$ div in ABS form $\sum \frac{1}{n}$ H.S. div.

Alternating H.S. converges ① $\frac{1}{n} > 0$ $n \geq 1$ ② $\lim_{n\to\infty} \frac{1}{n} = 0$ ③ $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2} < 0$ decr

d) $\sum_{1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$ RAT Test ABS CONV $\lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!}\,(n+1)\cancel{n!}}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{n!}\,\cancel{n!}} \right|$

$= \lim_{n\to\infty} \frac{\cancel{n+1}}{2\cancel{(n+1)}} \cdot \frac{\cancel{n}+1}{\cancel{2n}+1} \overset{\nearrow \frac{1}{2}}{} = \frac{1}{4} < 1 \Longrightarrow \boxed{\text{ABS CONV}}$ Ratio Test