Midterm #2

Answer Key

1\. a) A sequence is a list of numbers that come from a function whose domain is the set of ~~integers~~ integers $\geq$ some integer no.

"Ordered List of #'s following a rule" was acceptable.

A series is the infinite sum of a given sequence.
(which also tells how they are related)

Example: sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

b) $S_4 = \frac{7}{2} + \frac{7}{3} + \frac{7}{4} + \frac{7}{5}$

c) In the limit comparison test, you take $\lim_{x \to \infty} \frac{f(x)}{g(x)}$. There are three possible outcomes

i) $\lim_{x \to \infty} \frac{f}{g} = L \qquad 0 < L < \infty$

says either both f & g converge or both div.

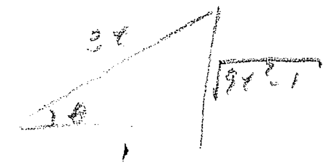
ii) $\lim_{x \to \infty} \frac{f}{g} = 0$ is no information

iii) $\lim_{x \to \infty} \frac{f}{g} = \infty$ is no information.

\# 2.a) $\displaystyle\int \frac{t\,dt}{\sqrt{9t^2-1}} = \left\{ \begin{array}{ll} t = \frac{1}{3}\sec\theta & 0<\theta<\frac{\pi}{2} \\ dt = \frac{1}{3}\sec\theta\tan\theta\, d\theta & \end{array} \right\}$

$$= \int \frac{\frac{1}{3}\sec\theta\, d\theta}{\sqrt{\sec^2\theta - 1}}\ \frac{1}{3}\sec\theta\tan\theta\, d\theta = \frac{1}{9}\int \frac{\sec^2\theta\tan\theta}{|\tan\theta|}\, d\theta$$

$$= \frac{1}{9}\int \sec^2\theta\, d\theta = \frac{1}{9}\tan\theta + C = \boxed{\frac{1}{9}\sqrt{9t^2-1} + C}$$



Alt: $u = 9t^2 - 1$, $du = 18t\, dt$

$$\int \frac{1}{18}\frac{du}{u^{1/2}} = \frac{1}{18}\left(2u^{1/2}\right) + c = \boxed{\frac{1}{9}\sqrt{9t^2-1} + C}$$

2.b) $\displaystyle\int e^{2x}\sin 3x\, dx = \left\{ \begin{array}{ll} u = \sin 3x & dv = e^{2x}dx \\ du = 3\cos 3x\, dx & v = \frac{1}{2}e^{2x} \end{array} \right\}$

$$I_1 = \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{2}\underbrace{\int \cos 3x\, e^{2x}\, dx}_{I_2}$$

$$I_2 = \left\{ \begin{array}{ll} u = \cos 3x & dv = e^{2x}dx \\ du = -3\sin 3x\, dx & v = \frac{1}{2}e^{2x} \end{array} \right\}$$

$$= \frac{1}{2}\cos 3x\, e^{2x} + \frac{3}{2}\int \sin 3x\, e^{2x}\, dx$$

$$I_1 = \frac{1}{2}\sin 3x\, e^{2x} - \frac{3}{4}\cos 3x\, e^{2x} - \frac{9}{4}\int \sin 3x\, e^{2x}\, dx$$

$$\left(1 + \frac{9}{4}\right) I_1 = \left(\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x\right)e^{2x} + C$$

$$I_1 = \frac{4}{13}\left(\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x\right)e^{2x} + C$$

$$\boxed{\text{Ans} = \frac{1}{13}\left(2\sin 3x - 3\cos 3x\right)e^{2x} + C}$$

2.c) $\int \frac{4t}{t^3+4t}\,dt = \int \frac{4t}{t(t^2+4)}\,dt = \int \frac{4}{t^2+4}\,dt \quad t \neq 0$

$= \int \frac{4}{4(\frac{1}{4}t^2+1)}\,dt = \left\{ \begin{array}{l} u = \frac{1}{2}t \\ du = \frac{1}{2}\,dt \end{array} \right\} = \int \frac{2\,du}{u^2+1}$

$= 2\tan^{-1}(u) + c = \boxed{2\tan^{-1}\left(\frac{t}{2}\right) + c}$

Alt: $\frac{4t}{t(t^2+4)} = \frac{A}{t} + \frac{Bt+c}{t^2+4}$

$\Rightarrow A(t^2+4) + Bt^2 + ct = 4t$

if $t = 0 \Rightarrow 4A = 0 \Rightarrow A = 0 \Rightarrow B = 0 \Rightarrow c = 4$

$\Rightarrow \int \frac{4}{t^2+4}\,dt = 4\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{t}{2}\right) + c = \boxed{2\tan^{-1}\left(\frac{t}{2}\right) + c}$

3. a) $a_n = \ln\left(\frac{n+\pi}{n}\right)^n$

$$\lim_{n\to\infty} \ln\left(\frac{n+\pi}{n}\right)^n = \ln\left(\lim_{n\to\infty}\left(\frac{n+\pi}{n}\right)^n\right) = \ln\left(\lim_{n\to\infty}\left(1+\frac{\pi}{n}\right)^n\right)$$

$$= \ln\left(e^{\pi}\right) = \pi \quad \therefore \boxed{\text{sequence converges to } \pi.}$$

by table 8.1 in book & formula sheet on back of exam

b) $a_n = n\left(1-\cos\left(\frac{1}{n}\right)\right)$ rewrite as $\dfrac{\left(1-\cos\left(\frac{1}{n}\right)\right)}{\frac{1}{n}}$

$$\lim_{n\to\infty} \frac{\left(1-\cos\left(\frac{1}{n}\right)\right)}{\frac{1}{n}} = \frac{0}{0} \text{ indeterminate form...}$$

L'Hop.

$$\longrightarrow = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)\cdot \cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}} = \lim_{n\to\infty} \sin\frac{1}{n} = 0$$

$\therefore$ sequence converges to 0.

4. a) $\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx$ Then $f(x) = \frac{1+\sin x}{x^2}$

* Is improper b/c ∞ in limits of integration

Know $-1 \le \sin x \le 1$

$\Rightarrow \quad 0 \le 1+\sin x \le 2$

$\Rightarrow \quad 0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$

So let $g(x) = \frac{2}{x^2}$ Then $\int_{\pi}^{\infty} \frac{2}{x^2}\,dx = 2\int_{\pi}^{\infty} \frac{1}{x^2}\,dx$

is a convergent p-integral w/ $p = 2 > 1$.

So since $\int_{\pi}^{\infty} g(x)\,dx$ converges and

$0 \le f(x) \le g(x)$

$0 \le \int_{\pi}^{\infty} f(x)\,dx \le \int_{\pi}^{\infty} g(x)\,dx$ so $\int_{\pi}^{\infty} f(x)$ converges

$\therefore \int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx$ converges by Direct Comparison Test w/ $\frac{2}{x^2}$.

4. b) $\int_0^2 \frac{ds}{1-s^2}$

* Is improper b/c the function goes to $\pm\infty$ at the interior point 1.

Split up the integral at the discontinuity.

$$= \underbrace{\int_0^1 \frac{1}{1-s^2}\,ds}_{} + \int_1^2 \frac{1}{1-s^2}\,ds.$$

First look at this piece

$\int_0^1 \frac{1}{1-s^2}\,ds$ can be easily integrated w/ partial fractions

$$\frac{1}{1-s^2} = \frac{1}{(1-s)(1+s)} = \frac{A}{1+s} + \frac{B}{1-s}$$

$$\Rightarrow \quad A(1-s) + B(1+s) = 1$$

$$\Rightarrow \quad A+B = 1$$

$$B-A = 0$$

$$\Rightarrow \quad A = B = \frac{1}{2}$$

Now $\int_0^1 \frac{1}{1-s^2}\,ds = \frac{1}{2}\int_0^1 \left[\frac{1}{1+s} + \frac{1}{1-s}\right] ds$

$$= \lim_{b\to 1^-} \frac{1}{2}\ln\left[\frac{1+s}{1-s}\right]_0^b = \frac{1}{2}\lim_{b\to 1^-} \ln\left[\frac{1+b}{1-b}\right]^{\infty} - \ln\left[\frac{1}{1}\right]^{0} = \infty$$

Since one piece of the original integral Diverges, The entire integral $\boxed{\text{Diverges}}$

5.a) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n} = \sum_{n=1}^{\infty} 3\left(\frac{-1}{4}\right)^n = \sum_{n=1}^{\infty} \frac{-3}{4}\left(\frac{-1}{4}\right)^{n-1}$

Since $|r| = \left|\frac{-1}{4}\right| < 1$, $\sum_{n=1}^{\infty} a_n \longrightarrow \frac{-3/4}{1 - -1/4} = \frac{-3}{5}$

b.) $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln(n)}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{\ln(n)} \overset{L.H.}{=} \lim_{n\to\infty} \frac{1/2\sqrt{n}}{1/n} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty$

$\therefore$ by the nth-Term Test for Divergence, $\sum_{n=2}^{\infty} b_n$ diverges.

c.) $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{n}{n^2+6}$

Let $f(x) = \frac{x}{x^2+6}$

$\int_1^{\infty} \frac{x}{x^2+6}\,dx = \left\{ \begin{matrix} u = x^2+6 \\ du = 2x\,dx \end{matrix} \right\} = \int_7^{\infty} \frac{1}{2u}\,du$

$= \lim_{b\to\infty} \int_7^b \frac{1}{2u}\,du = \lim_{b\to\infty} \frac{1}{2}\ln|u| \Big|_7^b = \lim_{b\to\infty} \frac{1}{2}(\ln b - \ln 7)$

$= \infty$

$\therefore \sum_{n=1}^{\infty} c_n$ diverges since $\int_1^{\infty} f(x)\,dx$ diverges.