Solutions : Midterm 3

1.a) Since the Taylor series $f(x) = e^x$ centered at 0 converges for all x, let

$$g(x) = e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

Then $g(1) = e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

Notice that,

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n!} = -x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + R_5(x)$$

where $R_5(x) = \frac{1}{(5+1)!} c^6$ for $0 < c < x$

$\therefore |R_5(1)| \le \frac{1}{6!}$

b.) False. Consider $a_n = \frac{1}{n}$. The $\lim_{n\to\infty} a_n = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (harmonic series).

c.) False. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. Then $0 \le a_n \le b_n$ for all $n = 1, 2, \ldots$. $\sum_{n=1}^{\infty} b_n$ diverges (above) but $\sum_{n=1}^{\infty} a_n$ converges (p-test).

d.) False. Consider the alternating series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{1/2}}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{1/3}}$

Both converge but $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n^{5/6}}$ diverges.

2 a) $$\lim_{x\to 0} \frac{\ln(1+x^2)}{1-\cos x} = \lim_{x\to 0} \frac{\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots\right)}{1-\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}$$

$$= \lim_{x\to 0} \frac{x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots}{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots}$$

$$= \lim_{x\to 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \cdots}{\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots}$$

$$= \frac{1}{1/2!}$$

$$= 2!$$

$$\boxed{= 2}$$

b) $$\int_0^1 x\sin(x^3)\,dx = \int_0^1 x\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \frac{x^{27}}{9!} - \cdots\right)dx$$

$$= \int_0^1 \left(x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \frac{x^{22}}{7!} + \frac{x^{28}}{9!} + \cdots\right)dx$$

$$= \frac{x^5}{5} - \frac{x^{11}}{11\cdot 3!} + \frac{x^{17}}{17\cdot 5!} - \frac{x^{23}}{23\cdot 7!} + \frac{x^{29}}{29\cdot 9!} - \cdots \Bigg|_0^1$$

$$= \frac{1}{5} - \frac{1}{11\cdot 3!} + \frac{1}{17\cdot 5!} - \frac{1}{23\cdot 7!} + \frac{1}{29\cdot 9!} - \cdots$$

$$\boxed{= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(6n+5)(2n+1)!}}$$

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3. a) $\sum_{n=3}^{\infty} \frac{\ln n}{\ln(\ln n)}$

consider the inequality $e^{n^n} > n$

Then $\Rightarrow$ $\ln e^{n^n} > \ln n$

$n^n > \ln n$

$\ln n^n > \ln(\ln n)$

$n \ln n > \ln(\ln n)$

$\frac{\ln n}{\ln(\ln n)} > \frac{1}{n}$

So let $b_n = \frac{1}{n}$ then $a_n > b_n$ and $\sum b_n$ is the divergent harmonic series.

$\boxed{\Rightarrow \sum_{n=3}^{\infty} \frac{\ln n}{\ln(\ln n)} \text{ Diverges by Direct Comparison Test.}}$

b) $\sum_{n=1}^{\infty} \tan \frac{1}{n}$ Let $b_n = \frac{1}{n}$. Then $\sum b_n$ is the divergent harmonic series.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan \frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sec^2 \frac{1}{n} \cdot \frac{-1}{n^2}}{\frac{-1}{n^2}}$$

$= \lim_{n\to\infty} \sec^2 \frac{1}{n} = 1$. Since $0 < 1 < \infty$

and $\sum \frac{1}{n}$ diverges, $\sum \tan \frac{1}{n}$ also diverges by Limit Comparison Test.

3c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Consider $b_n = \dfrac{\sqrt{n}}{n^2} = \dfrac{1}{n^{3/2}}$

Then $\displaystyle\sum_{n=1}^{\infty} b_n$ is a convergent p-series.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{1/2}}{n^2+1} \cdot n^{3/2}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$ (dominance of powers: same dominant exponent in numerator & denominator, so look at coefficients...)

Since $0 < 1 < \infty$ and $\sum b_n$ converges,

$\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ Converges as well by Limit Comparison Test.

4.a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$, Let $u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Notice that

(1) $u_n \geq 0$

(2) $u_n \geq u_{n+1}$

$\left(\frac{1}{1+\sqrt{2}} \geq \frac{1}{\sqrt{2}+\sqrt{3}} \geq \frac{1}{\sqrt{3}+2} + \cdots \right)$

(3) $u_n \to 0$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges.

Let $b_n = \frac{1}{2\sqrt{n+1}} = \frac{1}{\sqrt{n+1}+\sqrt{n+1}} \leq \frac{1}{\sqrt{n}+\sqrt{n+1}} = a_n$

Since $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n+1}}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges conditionally

b.) $\sum_{n=1}^{\infty} (-10)^{-n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{10^n} = \sum_{n=1}^{\infty} a_n$. Notice that $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ is a geometric series which converges since $|r| = \frac{1}{10} < 1$. Therefore $\sum_{n=1}^{\infty} a_n$ converges absolutely.

4.c) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$ Let $u_n = \frac{n!}{2^n}$

Notice that $\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!} = \frac{n+1}{2}$

≥ 1 for $n = 1, 2, 3, \ldots$

$\therefore$ u_n is not a decreasing seq. and the alt. series <u>diverges</u>.

6.a) $f(x) = e^{-1/x^3}$, $a = 0$ (Maclaurin series)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for } |x| < \infty$$

$$e^{-1/x^3} = \sum_{n=0}^{\infty} \frac{(-1/x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{x^{3n}\, n!}$$

b.) $f(x) = \frac{1}{x^2} = x^{-2}$, $a = 1$

$$f'(x) = -2x^{-3}, \quad f''(x) = (-2)(-3)x^{-4}$$

$$f'''(x) = (-2)(-3)(-4)x^{-5}$$

$$\Rightarrow f^{(n)}(x) = (-1)^n (n+1)!\, x^{-(n+2)}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n f^{(n)}(a)}{n!} = \sum_{n=0}^{\infty} \frac{(x-1)^n (-1)^n (n+1)!}{n!\, a^{(n+2)}}$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$$

5. a) $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ In absolulate value the series becomes

$\sum_{n=1}^{\infty} \frac{|x^n|}{n^n}$.

Then $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{|x^{n+1}|}{(n+1)^{n+1}} \cdot \frac{n^n}{|x^n|}$

$= \lim_{n\to\infty} \frac{|x| \cdot n^n}{(n+1)^{n+1}} = |x| \cdot \lim_{n\to\infty} \frac{n^n}{(n+1)^n \cdot (n+1)}$

$= |x| \cdot \lim_{n\to\infty} \underbrace{\left(\frac{n}{n+1}\right)^n}_{\to \frac{1}{e}} \cdot \frac{1}{n+1} = \frac{|x|}{e} \lim_{n\to\infty} \underbrace{\left(\frac{1}{n+1}\right)}_{\to 0}$

$= \frac{|x|}{e} \cdot 0 = 0 = \rho < 1 \quad \forall x$ converges absolutely by ratio test for all $x \in \mathbb{R}$.

$\therefore$

- series converges absolutely for all $x \in \mathbb{R}$
- No conditional convergence
- No divergence.

5. b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+4)^n}{n3^n}$ in absolute value we get $\displaystyle\sum_{n=1}^{\infty} \frac{|(x+4)|^n}{n3^n}$.

Then $\displaystyle\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{|(x+4)^{n+1}|}{(n+1)3^{n+1}} \cdot \frac{n3^n}{|(x+4)^n|}$

$$= \lim_{n\to\infty} \frac{|x+4|}{3} \cdot \frac{n}{n+1}$$

$$= \frac{|x+4|}{3} \cdot \lim_{n\to\infty} \frac{n}{n+1} \to 1 = \frac{|x+4|}{3} = \rho < 1 \quad \text{when}$$

$$\frac{|x+4|}{3} < 1 \Rightarrow |x+4| < 3$$

$$\Rightarrow -3 < x+4 < 3$$

$\Rightarrow -7 < x < -1$ gives absolute convergence. by Ratio Test

Test endpoints $x = -7, -1$

$x = -7 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{-3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is the convergent alternating harmonic series, which is conditionally convergent.

$n = -1 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series.

$\therefore$ - Original series
- converges absolutely for $-7 < x < -1$
- Converges conditionally for $x = -7$
- Diverges for $x < -7$, $x \geq -1$

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5. c) $\sum_{n=1}^{\infty} \frac{(x-1)^{2n-2}}{(2n-1)!}$ in abs. value $\Rightarrow \sum_{n=1}^{\infty} \frac{|(x-1)^{2n-2}|}{(2n-1)!}$

Then $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{|(x-1)^{2(n+1)-2}|}{(2(n+1)-1)!} \cdot \frac{(2n-1)!}{|(x-1)^{2n-2}|}$

$= \lim_{n\to\infty} \frac{|(x-1)^{2n}|\ (2n-1)!}{(2n+1)!\ |(x-1)^{2n-2}|}$

$= \lim_{n\to\infty} \frac{|(x-1)^2|}{(2n+1)(2n)} = (x-1)^2 \cdot \lim_{n\to\infty} \frac{1}{4n^2+2n} = (x-1)^2 \cdot 0 = 0$

$= \rho < 1$

$\Rightarrow$ Absolutely Convergent for all x by ratio test.

$\therefore$ - Abs. Conv. $\forall x \in \mathbb{R}$

- No CC

- No Div.