Exam 2 Solutions

1a) $I = \int e^{2x} \sin(3x)\,dx$ Using tabular

u	dv
e^{2x} $\oplus$	$\sin(3x)$
$2e^{2x}$ $\ominus$	$-\frac{1}{3}\cos(3x)$
$4e^{2x}$	$-\frac{1}{9}\sin(3x)$

After two steps of integration by parts, will get

$$I = e^{2x}\left(\tfrac{-1}{3}\cos(3x)\right) - (2e^{2x})\left(\tfrac{-1}{9}\sin(3x)\right) + \int 4e^{2x}\left(\tfrac{-1}{9}\sin(3x)\right)dx$$

$$= \tfrac{-1}{3}e^{2x}\cos(3x) + \tfrac{2}{9}e^{2x}\sin(3x) - \tfrac{4}{9}I + c$$

Solving for I gives

$$\tfrac{13}{9} I = e^{2x}\left(\tfrac{2}{9}\sin(3x) - \tfrac{1}{3}\cos(3x)\right) + c$$

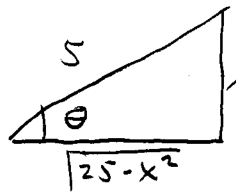
or

$$I = \tfrac{1}{13}e^{2x}\left(2\sin(3x) - 3\cos(3x)\right) + C \longleftarrow$$

1b) $I = \int \sqrt{25-x^2}\,dx$ sub is $\sin\theta = \frac{x}{5}$

so $dx = 5\cos\theta$

and $\sqrt{25-x^2} = 5|\cos\theta|$



$I = \int 5|\cos\theta| \cdot 5\cos\theta\,d\theta$ for $\left|\frac{x}{5}\right| \le 1$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

for which $\cos\theta \ge 0$

thus $|\cos\theta| = \cos\theta$

$$= \int 25\cos^2\theta\,d\theta$$

$$= \frac{25}{2}\int(1+\cos 2\theta)\,d\theta = \frac{25}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{25}{2}\left(\theta + \tfrac{1}{2}\cdot 2\sin\theta\cos\theta\right) + c$$

$$= \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) + \frac{25}{2}\cdot\frac{x}{5}\cdot\frac{\sqrt{25-x^2}}{5} + c$$

$$= \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) + \frac{1}{2}x\sqrt{25-x^2} + c \longleftarrow$$

1c) $I = \int_0^4 \frac{dx}{(x-2)^4} = \underbrace{\lim_{\varepsilon \to 0^+} \int_0^{2-\varepsilon} \frac{dx}{(x-2)^4}}_{A} + \underbrace{\lim_{\delta \to 0^+} \int_{2+\delta}^4 \frac{dx}{(x-2)^4}}_{B}$

If either A or B Div, then I DIV.

Look at A

$$\lim_{\varepsilon \to 0^+} \left(\frac{-1}{3}\right)(x-2)^{-3} \Big|_0^{2-\varepsilon} = \left(-\frac{1}{3}\right) \lim_{\varepsilon \to 0^+} \left[\frac{1}{(-\varepsilon)^3} - \frac{1}{(-2)^3}\right] \longrightarrow +\infty$$

so A Div. Thus I DIV.

1d) $I = \int \frac{2x^2+6x-5}{(x-2)(x^2+1)} dx$ Using partial fractions

$$\frac{2x^2+6x-5}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

or

$$2x^2+6x-5 = A(x^2+1) + (Bx+C)(x-2) \qquad (A)$$

$(A)\big|_{x=2}$ gives $15 = A \cdot 5$ $\quad A = 3$

$(A)\big|_{x=0}$ gives $-5 = A + -C \cdot 2$ so $C = \frac{A+5}{2} = 4$

$(A)\big|_{x=1}$ gives $3 = 2A + (B+C)(-1)$ so $B = -1$

so $I = \int \left(\frac{3}{x-2} + \frac{-x+4}{x^2+1}\right) dx = \int \left(\frac{3}{x-2} - \frac{1}{2}\frac{2x}{x^2+1} + \frac{4}{x^2+1}\right) dx$

$= 3 \ln|x-2| - \frac{1}{2}\ln|x^2+1| + 4\tan^{-1} x + C \longleftarrow$

(3/5)

2a) $I = \int_1^\infty \frac{x^2+5x+2}{x^4+4x^2+4}\,dx$ Do LCT with $\int_1^\infty \frac{dx}{x^2}$ which converges. P-Integral with $p=2>1$

Look at

$$\lim_{x\to\infty} \frac{\left(\frac{x^2+5x+2}{x^4+4x^2+4}\right)}{\left(\frac{1}{x^2}\right)} = 1$$

So Both converge ←

2b) $I = \int_3^\infty \frac{dx}{\tanh x} = \lim_{b\to\infty} \int_3^b \frac{\cosh x}{\sinh x}\,dx = \lim_{b\to\infty} \ln|\sinh x|\Big|_3^b$

$$= \lim_{b\to\infty}\left(\ln|\sinh b| - \ln|\sinh 3|\right)$$

$$= \lim_{b\to\infty}\left(\ln\left|\frac{e^b - e^{-b}}{2}\right| - \ln(\sinh 3)\right) \to \infty$$

∞

So I div. ←

3 a) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(1-\frac{2}{\sqrt{n}}\right)^n = L$ if it exists

so study $\lim\limits_{n\to\infty} n \cdot \ln\left(1-\frac{2}{\sqrt{n}}\right) = \ln L$

or $\lim\limits_{n\to\infty} \frac{\ln\left(1-\frac{2}{\sqrt{n}}\right)}{1/n} \overset{\text{L.H.}}{=} \lim\limits_{n\to\infty} \frac{\left(1-\frac{2}{\sqrt{n}}\right)^{-1}\left(\frac{+1}{n\sqrt{n}}\right)}{\left(-1/n^2\right)}$

$= \lim\limits_{n\to\infty} -\left(1-\frac{2}{\sqrt{n}}\right)\sqrt{n} \to -\infty = \ln L$

so $L \to e^{-\infty} = 0 \quad \therefore a_n$ converges

3 b) $a_n = \frac{2}{n!} + \frac{n(-1)^n}{n!}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(\frac{2}{n!}\right) \to 0 + \underbrace{\lim\limits_{n\to\infty} \frac{n(-1)^n}{n!} \to 0}_{\text{use Sand. Thm as below}} = 0 \quad \therefore$ conv.

$-\frac{n}{n!} \leq \frac{n(-1)^n}{n!} \leq \frac{n}{n!}$

so

$\underbrace{\lim\limits_{n\to\infty} -\left(\frac{n}{n!}\right)}_{0} \leq \lim \frac{n(-1)^n}{n!} \leq \underbrace{\lim\limits_{n\to\infty} \left(\frac{n}{n!}\right)}_{0}$

$\therefore \lim\limits_{n\to\infty} \frac{n(-1)^n}{n!} = 0$ also

5/5

4a) $$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{4^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{4}\left(\frac{-3}{4}\right)^n$$ reindex with $n=k+1$

$$= \sum_{k+1=1}^{\infty} \frac{1}{4}\left(\frac{-3}{4}\right)^{k+1} = \sum_{k=0}^{\infty} \frac{1}{4}\left(\frac{-3}{4}\right)\left(\frac{-3}{4}\right)^k$$

So this is a G.S. with $a = \frac{-3}{16}$ $r = \frac{-3}{4}$

Since $|r|<1$ this G.S. converges to

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{4^{n+1}} = \sum_{k=0}^{\infty} \left(\frac{-3}{16}\right)\left(\frac{-3}{4}\right)^k = \frac{a}{1-r} = \frac{(-3/16)}{1+3/4}$$

$$= -3/28$$

4b) $$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty}\left(\ln(n) - \ln(n+1)\right)$$

Look at N^{th} partial sum ...

$$S_N = \underbrace{(\ln(1) - \ln(2))}_{n=1} + \underbrace{(\ln(2) - \ln(3))}_{n=2} + \underbrace{(\ln(3) - \ln(4))}_{n=3}$$

$$+ \dots + \underbrace{(\ln(N-1) - \ln(N))}_{n=N-1} + \underbrace{(\ln N - \ln(N+1))}_{n=N}$$

so

$$S_N = -\ln(N+1)$$

thus $\lim_{N\to\infty} S_N \to -\infty$ $\therefore \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

DIV.