ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes, and crib sheets NOT permitted. No electronic devices may be used during the exam.

1. (15 points) Determine whether each statement is always TRUE, or FALSE (meaning not always true), assuming each summation goes from $n = 1$ to $n = \infty$. You do not need to justify your answers, but you must write the complete word TRUE or FALSE to receive credit.
   (a) If $\sum |a_n|$ converges, then $\sum a_n$ converges.
   (b) If $\sum |a_n|$ diverges, then $\sum a_n$ diverges.
   (c) If $\sum a_n$ converges, then $\sum a_n(-1)^n$ converges.
   (d) The Ratio Test can be used to determine whether $\sum \frac{1}{(n+1)^2}$ converges.
   (e) If $a_n \le b_n \le c_n$ and $\sum a_n$ converges and $\sum c_n$ diverges, then $\sum b_n$ converges.

2. (20 points) Determine whether the following infinite series converge absolutely, converge conditionally, or diverge. Be sure to justify your answers.
   (a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-e}{n+e}\right)^n$
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n^4}{n!}$

3. (15 points) Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n}}$.
   (a) Where is the series absolutely convergent?
   (b) Where is the series conditionally convergent?
   (c) Where is the series divergent?

4. (20 points) Consider the function $f(x) = \sin(x)$.
   (a) Starting from the definition, derive the Taylor Series of $f(x)$ with a center of $a = \pi$.
   (b) Use the first two non-zero terms of your series to estimate the value of $\sin(x)$ if $x = 183$ degrees. Note that 180 degrees equals π radians and 3 degrees equals $\pi/60$ radians. You may leave your answer in terms of fractions, factorials, etc.
   (c) Determine an upper bound for the magnitude of the error associated with your estimate in part (b).
   (d) Finally, is your estimate in part (b) too large, too small, or if it is not possible to determine, state "Not enough information." Explain your reasoning.

5. (15 points) Consider a function $f(x)$. If you were to calculate the third derivative of f with respect to x, you would find that $f'''(x) = \frac{4}{11}(1+x)^{\frac{1}{4}-3}$. Estimate the error in the approximation $f(-\frac{1}{2}) \approx P_3(-\frac{1}{2})$, where $P_3(x)$ is computed with a center of $a = 0$.

6. (15 points) Alvin the ant likes to go for morning walks before all the other ants are awake. In fact, each morning he walks along the path described by the function $y(x) = \frac{x^3}{3}$ for $0 \le x \le 0.1$. One day, having just finished his Calculus II class, he realized that he can use a Maclaurin series to estimate how far he travels during his morning walks.
   (a) What are the first three non-zero terms in the Maclaurin series for the integrand of the arc length integral that Alvin would use in his calculations.
   (b) After Alvin integrates the terms in part (a), what is his final approximation for the distance traveled.

BE SURE TO LOOK AT THE INFORMATION ON THE BACK

A short table of integrals. In the following, $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{for } a > 0$

2. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \text{for } u^2 < a^2$

3. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{for } u > a > 0$

4. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

5. $\displaystyle\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 > a^2 \end{cases}$

6. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C \qquad \text{for } u \neq 0$

7. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C \qquad \text{for } 0 < u < a$

8. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad \text{for } u^2 > a^2$

Some circular and hyperbolic trig identities.

1. $\cos^2 x + \sin^2 x = 1$
2. $\cos^2 x = \dfrac{1+\cos(2x)}{2}$
3. $\sin^2 x = \dfrac{1-\cos(2x)}{2}$
4. $\cosh^2 x - \sinh^2 x = 1$
5. $\cosh^2 x = \dfrac{\cosh(2x)+1}{2}$
6. $\sinh^2 x = \dfrac{\cosh(2x)-1}{2}$

In formulas (3)–(6), x remains fixed as $n \to \infty$.

1. $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\displaystyle\lim_{n\to\infty} x^{1/n} = 1, \quad x > 0$
4. $\displaystyle\lim_{n\to\infty} x^n = 0, \quad |x| < 1$
5. $\displaystyle\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x, \quad |x| < \infty$
6. $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0, \quad |x| < \infty$

Here are some common Maclaurin Series and the values of x for which they converge.

1. $\displaystyle\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$

2. $\displaystyle\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-x)^n, \qquad |x| < 1$

3. $\displaystyle e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$

4. $\displaystyle\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$

5. $\displaystyle\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$

6. $\displaystyle\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \leq 1$

7. $\displaystyle\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \leq 1$

8. $\displaystyle(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}x^n + \cdots$
   $\displaystyle= \sum_{n=0}^{\infty} \binom{m}{n} x^n, \qquad |x| < 1$

Arc length $\displaystyle L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$