APPM 1360 Exam 1 ~~Fall~~ Spr 2009

Region for Problems 1 - 3

$y=\sqrt{x}$; $y=\frac{x^2}{8}$

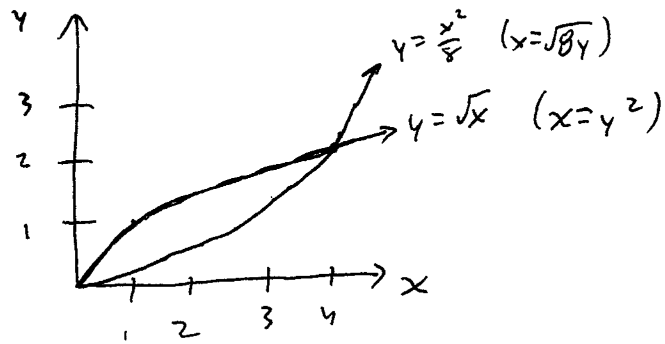
Intersection:

$\sqrt{x}=\frac{x^2}{8} \Rightarrow \frac{x^2}{8}-\sqrt{x}=0$

$\sqrt{x}\left(\frac{x^{3/2}}{8}-1\right)=0$

$\sqrt{x}=0$ $\quad x^{3/2}=8$

$\Rightarrow x=0$ $\quad \Rightarrow x=8^{2/3}=2^2=4$



① a) Perimeter = Arclength of $y=\sqrt{x}$ + Arclength of $y=\frac{x^2}{8}$ (from $x=0$ to $x=4$)

$$P=\int_0^4\sqrt{1+\left(\frac{d}{dx}[\sqrt{x}]\right)^2}\,dx+\int_0^4\sqrt{1+\left(\frac{d}{dx}\left[\frac{x^2}{8}\right]\right)^2}\,dx$$

$$=\int_0^4\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx+\int_0^4\sqrt{1+\left(\frac{x}{4}\right)^2}\,dx$$

$$=\int_0^4\sqrt{1+\frac{1}{4x}}\,dx+\int_0^4\sqrt{1+\frac{x^2}{16}}\,dx$$

Note that $\sqrt{1+\frac{1}{4x}}$ is not continuous at $x=0$, so you should find arc-length of $x=y^2$ instead:

$$\int_0^4\sqrt{1+\left(\frac{d}{dx}[\sqrt{x}]\right)^2}\,dx=\int_0^2\sqrt{1+\left(\frac{d}{dy}[y^2]\right)}\,dy$$

$$=\int_0^2\sqrt{1+(2y)^2}\,dy$$

$$=\int_0^2\sqrt{1+4y^2}\,dy$$

So $\boxed{P=\int_0^2\sqrt{1+4y^2}\,dy+\int_0^4\sqrt{1+\frac{x^2}{16}}\,dx}$

b) Area $=\int_0^4\left(\sqrt{x}-\frac{x^2}{8}\right)dx$

$$=\frac{2}{3}x^{3/2}-\frac{x^3}{24}\Big|_0^4$$

$$=\frac{2}{3}(8)-\frac{64}{24}$$

$$=\frac{16}{3}-\frac{8}{3}=\boxed{\frac{8}{3}}$$

(2) $M = \int_0^4 \underbrace{\delta\left(\sqrt{x} - \frac{x^2}{8}\right)dx}_{dm}$

$= \delta \int_0^4 \left(\sqrt{x} - \frac{x^2}{8}\right)dx = \boxed{\frac{8\delta}{3}}$ (From 1b)

$M_x = \int_0^4 \underbrace{\frac{1}{2}\left(\sqrt{x} + \frac{x^2}{8}\right)}_{\tilde{y}} \underbrace{\delta\left(\sqrt{x} - \frac{x^2}{8}\right)dx}_{dm}$

$= \frac{\delta}{2}\int_0^4 \left(x - \frac{x^4}{64}\right)dx$

$= \frac{\delta}{2}\left(\frac{x^2}{2} - \frac{x^5}{5 \cdot 64}\right)\Big|_0^4$

$= \frac{\delta}{2}\left(8 - \frac{16}{5}\right)$ Helpful hint: $64 = 4^3$, so $\frac{4^5}{64} = \frac{4^5}{4^3} = 4^2 = 16$

$= \frac{\delta}{2}\left(\frac{24}{5}\right) = \boxed{\frac{12\delta}{5}}$

$M_y = \int_0^4 \underbrace{x}_{\tilde{x}} \underbrace{\delta\left(\sqrt{x} - \frac{x^2}{8}\right)dx}_{dm}$

$= \delta \int_0^4 \left(x^{3/2} - \frac{x^3}{8}\right)dx$

$= \delta\left(\frac{2x^{5/2}}{5} - \frac{x^4}{32}\right)\Big|_0^4$

$= \delta\left(\frac{64}{5} - 8\right)$ Helpful hints: $4^{5/2} = (\sqrt{4})^5 = 2^5$

$32 = 2 \cdot 16 = 2 \cdot 4^2 \Rightarrow \frac{4^4}{32} = \frac{4^2}{2} = 8$

$= \boxed{\frac{24\delta}{5}}$

$\bar{x} = \frac{M_y}{M} = \frac{\frac{24\delta}{5}}{\frac{8\delta}{3}} = \boxed{\frac{9}{5}}$ $\bar{y} = \frac{M_x}{M} = \frac{\frac{12\delta}{5}}{\frac{8\delta}{3}} = \boxed{\frac{9}{10}}$

$\boxed{\text{C.O.M.: } \left(\frac{9}{5}, \frac{9}{10}\right)}$

(3) a) Washer Method + Revolution around vertical line

$\Rightarrow$ Integrate in y!

$R(y) = \sqrt{8y}$, $r(y) = y^2$

$V = \int_0^2 \pi\left((\sqrt{8y})^2 - (y^2)^2\right)dy$

$\boxed{V = \pi \int_0^2 (8y - y^4)\,dy}$

b) Shell Method + Horizontal Line

$\Rightarrow$ Integrate in y!

$r(y) = (y - (-2)) = y + 2$

$h(y) = \sqrt{8y} - y^2$

$\boxed{V = \int_0^2 2\pi (y+2)(\sqrt{8y} - y^2)\,dy}$

(4) a) $\int_1^{e^2} \ln x\,dx = x \ln x \Big|_1^{e^2} - \int_1^{e^2} x\left(\frac{1}{x}\right)dx$

$\left\{\begin{matrix} u = \ln x & dv = dx \\ du = \frac{1}{x}dx & v = x \end{matrix}\right\}$ $= e^2 \ln e^2 - \ln 1 - \int_1^{e^2} dx$

$= 2e^2 - (e^2 - 1)$

$= \boxed{e^2 + 1}$

b) $2\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos^2 x}{\cos x}\,dx = 2\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}dx + 2\int_0^{\frac{\pi}{4}} \cos x\,dx$

$\{u = \cos x \quad du = -\sin x\,dx\}$

$= -2\int_1^{\sqrt{2}/2} \frac{1}{u}du + 2\sin x \Big|_0^{\frac{\pi}{4}}$

$= -2\ln u\Big|_1^{\sqrt{2}/2} + 2\left(\frac{\sqrt{2}}{2} - 0\right)$

$= -2\ln\left(\frac{\sqrt{2}}{2}\right) + \sqrt{2} = \boxed{\ln 2 + \sqrt{2}}$

④ c) $\displaystyle\int \frac{1}{\cosh x+1}\,dx = \int \frac{1}{\cosh x+1}\cdot\frac{(\cosh x-1)}{(\cosh x-1)}\,dx$

$$= \int \frac{\cosh x-1}{\cosh^2 x-1}\,dx = \int \frac{\cosh x-1}{\sinh^2 x}\,dx$$

$$= \int\left(\frac{\cosh x}{\sinh^2 x} - \frac{1}{\sinh^2 x}\right)dx$$

$$= \int(\operatorname{csch} x\coth x - \operatorname{csch}^2 x)\,dx$$

$$\boxed{= -\operatorname{csch} x + \coth x + C}$$

⑤ Separable and First-Order Linear! (7.3, #43)

Sep of Variables

$$\frac{dy}{dx} = \frac{2y+2}{x^2+2x} \Rightarrow \int\frac{dy}{y+1} = \int\frac{2dx}{x^2+2x}$$

Partial Fractions!

$$\frac{2}{x^2+2x} = \frac{2}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

So $2 = A(x+2) + Bx$

$x=0 \Rightarrow 2 = 2A + 0B$ so $A=1$

$x=-2 \Rightarrow 2 = 0A - 2B$ so $B=-1$

$$\int\frac{2dx}{x^2+2x} = \int\left(\frac{1}{x} - \frac{1}{x+2}\right)dx = \ln|x| - \ln|x+2| + C$$

$$= \ln\left(\frac{x}{x+2}\right) + C$$

$\displaystyle\int\frac{dy}{y+1} = \int\frac{2dx}{x^2+2x}$, so

$$\ln|y+1| = \ln\left|\frac{x}{x+2}\right| + C \Rightarrow y+1 = \frac{Cx}{x+2}$$

$y(1)=0$, so $1 = \frac{C}{3} \Rightarrow C=3$

$$\boxed{y(x) = \frac{3x}{x+2} - 1}$$

Integrating Factor:

$$v(x) = e^{\int P(x)dx}$$

$$y(x) = \frac{1}{v(x)}\int v(x)Q(x)\,dx$$

$$\frac{dy}{dx} - \frac{2}{x^2+2x}\,y = \frac{2}{x^2+2x}$$

$$v(x) = e^{\int\frac{-2}{x^2+2x}dx} = e^{-\int\frac{2}{x^2+2x}dx}$$

$$= e^{-\ln\left(\frac{x}{x+2}\right)}$$ (see Sep. of Variables for integration)

$$= \frac{x+2}{x}$$

$$y(x) = \frac{x}{x+2}\int\frac{x+2}{x}\cdot\frac{2}{x^2+2x}\,dx$$

$$= \frac{2x}{x+2}\int\frac{x+2}{x(x+2)x}\,dx$$

$$= \frac{2x}{x+2}\int\frac{dx}{x^2}\,dx$$

$$= \frac{-2x}{x+2}\left(\frac{1}{x} + C\right)$$

$y(1)=0 \Rightarrow 0 = \frac{-2}{3}(1+C)$, so $C=-1$

$$y(x) = \frac{-2x}{x+2}\left(\frac{1}{x} - 1\right)$$

$$= \frac{-2x}{x+2}\left(\frac{1-x}{x}\right) = \boxed{\frac{2x-2}{x+2}}$$

Note: $\displaystyle\frac{3x}{x+2} - 1 = \frac{3x}{x+2} - \frac{(x+2)}{x+2}$

$= \dfrac{2x-2}{x+2}$ so answers are equivalent.