ATPM 1360 - EXAM 3 - SPRING 2009

1(a) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Note $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series

so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is abs. conv by Abs. Conv. Test.

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

Note ① $\frac{1}{\ln(n)} > 0$ for $n \geq 2$

② $\frac{1}{\ln(n+1)} \leq \frac{1}{\ln(n)}$ for $n \geq 2$

③ $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$

so $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges by AST.

Now $\sum_{n=2}^{\infty} \frac{1}{\ln(n)} > \sum_{n=2}^{\infty} \frac{1}{n}$ ← div. Harmonic series

so $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges by D.C.T.,

so $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ is cond. conv.

2(a) $f(x) = \ln(1+x)$, $f(0) = 0$

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = \frac{-1}{(1+x)^2}$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

$$f^{(4)}(x) = \frac{-2\cdot 3}{(1+x)^4}, \text{ so } f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}, \quad n \geq 1$$

$$f^{(n)}(0) = \frac{(-1)^{n+1}(n-1)!}{1^n} = (-1)^{n+1}(n-1)!$$

so $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$= x - x^2/2 + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

(b) $$\int_0^{0.1} \ln(1+x)\,dx \approx \int_0^{0.1} \left(x - x^2/2 + x^3/3\right) dx$$

$$= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} \Big|_0^{0.1} = \frac{(0.1)^2}{2} - \frac{(0.1)^3}{6} + \frac{(0.1)^4}{12}$$

(c) $$|\text{error}| \leq \int_0^{0.1} \left|\frac{-x^4}{4}\right| dx = \frac{x^5}{20}\Big|_0^{0.1} = \frac{(0.1)^5}{20}$$

(using Alternating Series Est. Thm.)

(d) Error has same sign as $\int_0^{0.1} -\frac{x^4}{4}\,dx = \frac{-(0.1)^5}{20} < 0$

so we have an <u>overestimate</u>.

3. (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)\,2^{n+1}}\cdot\frac{n\,2^n}{(x+2)^n}\right|$

$= \lim_{n\to\infty}\left|\frac{(x+2)\,n}{(n+1)\,2}\right| = \frac{|x+2|}{2}\lim_{n\to\infty}\underbrace{\frac{n}{n+1}}_{\to 1} = \frac{|x+2|}{2}$

$\frac{|x+2|}{2} < 1 \Rightarrow |x+2| < 2 \Rightarrow$ R.O.C $= 2$

(b) $|x+2| < 2 \Rightarrow -2 < x+2 < 2 \Rightarrow -4 < x < 0$ Abs. Conv on (after checking endpts, see (c))

(c) $x = -4 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-2)^n}{n\,2^n} = \sum_{n=1}^{\infty}\frac{(-1\cdot -2)^n(-1)}{n\,2^n} = \underbrace{\sum_{n=1}^{\infty}\frac{-1}{n}}$

divergent since $\sum 1/n$ is divergent Harmonic Series

$x = 0 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2)^n}{n\,2^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ Alt Series Test

① $u_n = 1/n > 0$ for $n \geq 1$

② $n+1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n}$ for $n \geq 1$

③ $\lim_{n\to\infty}\frac{1}{n} = 0$

So series is Cond. Convergent

$\left(\sum\left|\frac{(-1)^{n+1}}{n}\right| - \sum\frac{1}{n}\ \text{H.S.}\right)$

Cond. Conv when $x = 0$.

(d) Divergent on $(-\infty, -4] \cup (0, \infty)$

4. (a) $e^x = \sum_{x=0}^{\infty} \frac{x^n}{n!} \Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{2x(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{n!}$

(b) Note,
$$\int_0^1 2xe^{x^2} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{n!} dx$$

and,

- $$\int_0^1 2xe^{x^2} = e^{x^2}\Big|_0^1 = e^1 - e^0 = e-1$$

- $$\int_0^1 \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{n!} dx = \sum_{n=0}^{\infty} \frac{2}{n!} \frac{x^{2n+2}}{2n+2}\Bigg|_0^1$$
$$= \sum_{n=0}^{\infty} \frac{2}{(2n+2)n!} = \sum_{n=0}^{\infty} \frac{1}{(n+1)n!}$$

so, $$\sum_{n=0}^{\infty} \frac{1}{(n+1)n!} = e-1.$$

5 (a) $$\lim_{x\to 0} \frac{x - \tan^{-1}(x)}{x^3} \overset{\text{"0/0"}}{=} \lim_{x\to 0} \frac{x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right)}{x^3}$$
$$= \lim_{x\to 0} \frac{1}{3} + \underbrace{\frac{x^2}{5} \cdots}_{\to 0} = \frac{1}{3}$$

(b) $$\left(1+\frac{1}{x}\right)^{1/2} = 1 + \sum_{k=1}^{3} \binom{1/2}{k}\left(\frac{1}{x}\right)^k$$
$$= 1 + \binom{1/2}{1}\frac{1}{x} + \binom{1/2}{2}\frac{1}{x^2} + \binom{1/2}{3}\frac{1}{x^3}$$
$$= 1 + \frac{1}{2x} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{x^2} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{x^3} = 1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3}$$

6. (a) $c < 9 \Rightarrow c - 9 < 0$ and $16 - c > 0$ so hyperbola

(b) $9 < c < 16 \Rightarrow c - 9 > 0$ & $16 - c > 0$, so ellipse

Note, if $c - 9 = 16 - c \Rightarrow 2c = 25$

$c = 25/2$

we have a circle.

(c) $c > 16 \Rightarrow c - 9 > 0 \Rightarrow 16 - c < 0 \Rightarrow$ hyperbola.