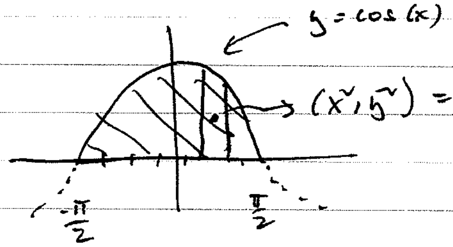
APPM 1360 - SUM - 09
TEST 1 - Soln's.

1.



$(\tilde{x}, \tilde{y}) = \left(x, \frac{\cos(x) + 0}{2}\right) = \left(x, \frac{\cos(x)}{2}\right)$

By symmetry $\bar{x} = 0$

$$\bar{y} = \frac{M_x}{m}$$

$$m = \int_{-\pi/2}^{\pi/2} \delta \cdot [\cos(x) - 0]\, dx = \int_{-\pi/2}^{\pi/2} \delta \cos(x)\, dx \left(= \int dm \right)$$

$$= \delta \sin(x) \Big|_{-\pi/2}^{\pi/2} = \delta\,(1 - (-1)) = 2\delta$$

and

$$M_x = \int \tilde{y}\, dm = \int_{-\pi/2}^{\pi/2} \frac{\cos(x)}{2} \cdot \delta \cdot \cos(x)\, dx$$

$$= \frac{\delta}{2} \int_{-\pi/2}^{\pi/2} \cos^2(x)\, dx$$

$$= \frac{\delta}{2} \int_{-\pi/2}^{\pi/2} \frac{1 + \cos(2x)}{2}\, dx$$

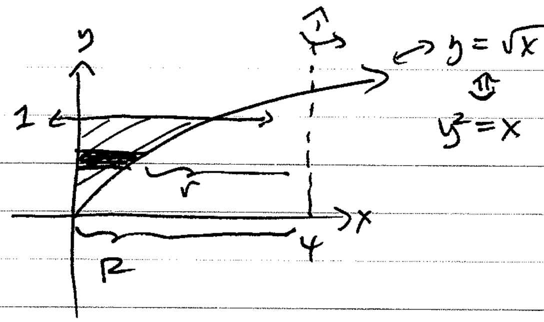
$$= \frac{\delta}{4} \left[x + \frac{\sin(2x)}{2} \right] \Big|_{-\pi/2}^{\pi/2}$$

$$= \delta/4 \left[\pi/2 - (-\pi/2) + 0 \right] = \frac{\delta \pi}{4}$$

so $\bar{y} = M_x/m = \frac{\delta \pi/4}{2\delta} = \frac{\pi}{8}$

so centroid $(\bar{x}, \bar{y}) = \underline{(0, \pi/8)}$.

2.



Disk Method:

$$\Delta V = \pi \left[R^2 - r^2 \right] \Delta y$$

$$\left.\begin{array}{l} R = 4 - 0 = 4 \\ r = 4 - y^2 \end{array}\right\} \Rightarrow \Delta V = \pi \left[4^2 - (4-y^2)^2 \right] \Delta y$$

$$= \pi \left[16 - (16 - 8y^2 + y^4) \right] \Delta y$$

$$= \pi \left(8y^2 - y^4 \right) \Delta y$$

So, $$V = \int_0^1 \pi (8y^2 - y^4)\, dy$$

$$= \pi \left(\frac{8y^3}{3} - \frac{y^5}{5} \right) \Bigg|_0^1$$

$$= \pi \left(\frac{8}{3} - \frac{1}{5} \right) = \pi \left(\frac{40}{15} - \frac{3}{15} \right)$$

$$= \underline{\underline{37\pi/15}}$$

3. (a) $\cosh(x) \frac{dy}{dx} + \sinh(x)\, y = e^{-x}$

$$\frac{dy}{dx} + \underbrace{\frac{\sinh(x)}{\cosh(x)}}_{P(x)} y = \underbrace{\frac{e^{-x}}{\cosh(x)}}_{Q(x)}$$

Integrating factor: $V(x) = e^{\int \frac{\sinh(x)}{\cosh(x)} dx} = e^{\int 1/u \, du} = e^{\ln(u)} = u = \cosh(x)$

$u = \cosh(x)$
$du = \sinh(x) dx$

So, $$y = \frac{1}{\cosh(x)} \int \frac{e^{-x}}{\cosh(x)} \cdot \cosh(x)\, dx = \frac{1}{\cosh(x)} \int e^{-x} dx$$

$$= \frac{1}{\cos(x)} \left[-e^{-x} + C \right]$$

$$= \frac{-e^{-x}}{\cosh(x)} + \frac{C}{\cosh(x)}.$$

3 (b) $\frac{dy}{dx} = \frac{2x^2+1}{xe^y}$

$$\frac{dy}{dx} = \frac{2x^2+1}{x} \cdot \frac{1}{e^y}$$

$$e^y \frac{dy}{dx} = \frac{2x^2+1}{x} \Rightarrow \int e^y \, dy = \int \frac{2x^2+1}{x} \, dx$$

$$e^y + C_1 = \int 2x + \frac{1}{x} \, dx$$

$$e^y + C_1 = x^2 + \ln|x| + C_2$$

$$e^y = x^2 + \ln|x| + C_3$$

$$\underline{\underline{y = \ln(x^2 + \ln|x| + C_3)}}$$

4 (a)

$$L = \int_c^d \sqrt{1+g'(y)^2} \, dy, \text{ here } g(y) = 2\sqrt{4-y}, \quad 1 \le y \le 4$$

$$g'(y) = 2 \cdot \tfrac{1}{2} (4-y)^{-1/2} \cdot -1 = -(4-y)^{-1/2}$$

$$(g'(y))^2 = (4-y)^{-1}$$

so $1 + g'(y)^2 = 1 + \frac{1}{4-y} = \frac{4-y}{4-y} + \frac{1}{4-y} = \frac{5-y}{4-y}$

~~so~~ $\cancel{L = \int_1^4 1 + 5 - y}$

so $\underline{\underline{L = \int_1^4 \sqrt{\frac{5-y}{4-y}} \, dy}}$.

4 (b) $SA = \int_c^d 2\pi\, g(y) \sqrt{1+g'(y)^2}\, dy$, $g(y) = 2\sqrt{4-y}$

$= \int_1^4 2\pi \cdot 2\sqrt{4-y} \cdot \sqrt{\frac{5-y}{4-y}}\, dy$

$= 4\pi \int_1^4 \sqrt{5-y}\, dy = 4\pi \cdot -\frac{2}{3} \cdot (5-y)^{3/2} \Big|_1^4$

$\boxed{\begin{array}{l} u = 5-y \\ du = -dy \end{array}}$

$= -\frac{8\pi}{3}\left[1^{3/2} - 4^{3/2}\right]$

$= -\frac{8\pi}{3}\left[1-8\right] = \frac{56\pi}{3}$

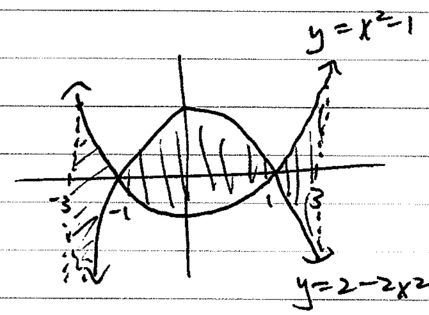
5 (a)

$y = x^2 - 1$, $y = 2 - 2x^2$

Pt. of Intersection: $x^2 - 1 = 2 - 2x^2$

$3x^2 = 3$

$x^2 = 1 \Rightarrow x = \pm 1$



(b)

$\text{Area} = \int_{-3}^{-1} \left[(x^2-1) - (2-2x^2)\right] dx + \int_{-1}^{1} \left[(2-2x^2) - (x^2-1)\right] dx$

$+ \int_1^3 \left[(x^2-1) - (2-2x^2)\right] dx$

$= \int_{-3}^{-1} (3x^2 - 3)\, dx + \int_{-1}^{1} (3 - 3x^2)\, dx + \int_1^3 (3x^2-3)\, dx$

OR by symmetry,

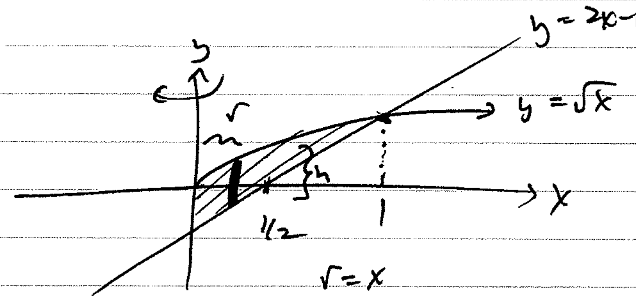
$A = 2\left[\int_0^1 (3-3x^2)\, dx + \int_1^3 (3x^2-3)\, dx\right]$

Shell Method

6 (a) $y = 2x - 1,\ y = \sqrt{x}$

$\Delta V = 2\pi \cdot r \cdot h \cdot \Delta x$

Pt. of Intersection:



$2x - 1 = \sqrt{x}$

$(2x-1)^2 = x$

$4x^2 - 4x + 1 = x$

$4x^2 - 5x + 1 = 0 \Rightarrow (4x-1)(x-1) = 0$

~~(4x²+1)(~~

$x = \frac{1}{4}, 1$

$r = x$

$h = \sqrt{x} - (2x-1)$

$= \sqrt{x} - 2x + 1$

so,

6 (b) $V = \int_0^1 2\pi \cdot x \cdot (\sqrt{x} - 2x + 1)\, dx$

$= 2\pi \int_0^1 (x^{3/2} - 2x^2 + x)\, dx$