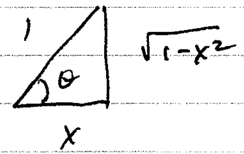
APPM 1360 SUMMER '09
TEST 2 soln's.

1(a) $$\int \frac{\sqrt{1-x^2}}{x^2}\,dx = \int \frac{\sin\theta \cdot -\sin\theta\, d\theta}{\cos^2\theta}$$



$\sin\theta = \sqrt{1-x^2}$

$\cos\theta = x \rightsquigarrow x^2 = \cos^2\theta$

$-\sin\theta\, d\theta = dx$

$\theta = \cos^{-1}(x)$

$$= -\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$$

$$= -\int \tan^2\theta\, d\theta$$

$$= -\int \left(\sec^2(\theta) - 1\right) d\theta$$

$$= \int 1 - \sec^2(\theta)\, d\theta$$

$$= \theta - \tan\theta + C$$

$$= \cos^{-1}(x) - \frac{\sqrt{1-x^2}}{x^2} + C$$

(b) $$\int_0^1 \frac{dx}{x^{1/2}+x^{3/2}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x^{1/2}+x^{3/2}}$$

Now note, $$\int \frac{dx}{x^{1/2}+x^{3/2}} = \underbrace{\int \frac{dx}{x^{1/2}(1+x)}}_{\text{let } u = x^{1/2} \Rightarrow x = u^2} = \int \frac{2\,du}{1+u^2} = 2\tan^{-1}(u) + C$$

$du = \frac{1}{2x^{1/2}}\,dx$

so $2\,du = \frac{dx}{x^{1/2}}$

$= 2\tan^{-1}(\sqrt{x}) + C$

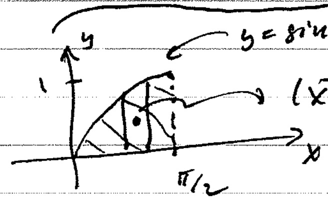
So,
$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{x^{1/2}+x^{3/2}} = \lim_{a\to 0^+} 2\tan^{-1}(\sqrt{x})\Big|_a^1$$

$$= \lim_{a\to 0^+} 2\left(\tan^{-1}(1) - \tan^{-1}(a)\right)$$

$\lim_{a\to 0^+} \tan^{-1}(a) = 0$

$$= 2\left(\pi/4 - 0\right) = 2\pi/4 = \pi/2$$

2. $M_y = \int x^2\, dm = \int_0^{\pi/2} x \cdot x\sin(x)\,dx = \int_0^{\pi/2} x^2 \sin(x)\,dx$



$(\tilde{x}, \tilde{y}) = \left(x, \frac{\sin(x)}{2}\right)$

Note $dm = \delta \cdot \sin(x)\,\Delta x$
$= x\sin(x)\,\Delta x$

Now $\int x^2 \sin(x)\,dx$ use I.B.P. twice:

let $u = x^2 \quad dv = \sin(x)\,dx$
$du = 2x\,dx \quad v = -\cos(x)$

$$\int x^2 \sin(x)\,dx = -x^2\cos(x) + \int 2x\cos(x)\,dx$$

$u = x \quad dv = \cos(x)\,dx$
$du = dx \quad v = \sin(x)\,dx$

so $\int x^2 \sin(x)\,dx = -x^2\cos(x) + 2\left[x\sin(x) - \int \sin(x)\,dx\right]$

$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$

so,

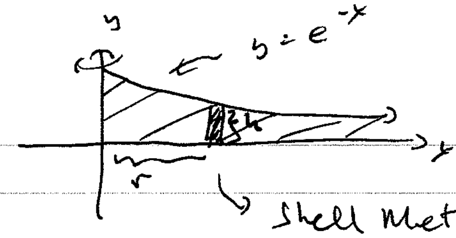
$$M_y = \int_0^{\pi/2} x^2 \sin(x)\,dx = -x^2\cos(x) + 2x\sin(x) + 2\cos(x)\Big|_0^{\pi/2}$$

$= -0 + 2\cdot\frac{\pi}{2}\sin(\pi/2) + 2\cdot 0 - [0 + 0 + 2]$

$\cancel{= \pi + 2}$

$= \pi - 2$

3.



Shell Method

$$\Delta V = 2\pi r h \, \Delta x = 2\pi x e^{-x} \Delta x$$

$$r = x$$
$$h = e^{-x}$$

So

$$V = \int_0^\infty 2\pi x e^{-x} dx$$

$$= \lim_{b\to\infty} \int_0^b 2\pi x e^{-x} dx.$$

Note, $\int x e^{-x} dx$ use IBP.

$$u = x \quad dv = e^{-x} dx$$
$$du = dx \quad v = -e^{-x} dx$$

$$\int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx$$

$$= -xe^{-x} - e^{-x} + C = -e^{-x}(x+1) + C$$

$$= \frac{-(x+1)}{e^x} + C$$

So,

$$V = \lim_{b\to\infty} \int_0^b 2\pi x e^{-x} dx = \lim_{b\to\infty} -\frac{2\pi(x+1)}{e^x}\Big|_0^b$$

$$= \lim_{b\to\infty} -2\pi\left[\frac{b+1}{e^b} - 1\right]$$

$$= -2\pi[0-1] = 2\pi$$

Note by L'Hospital

$$\lim_{b\to\infty} \frac{b+1}{e^b} \overset{LH}{=} \lim_{b\to\infty} \frac{1}{e^b} = 0$$

4(a) $a_n = n\left(1-\cos\left(\frac{1}{n}\right)\right)$

$$\lim_{n\to\infty} n\left(1-\cos(1/n)\right) \overset{``\infty\cdot 0"}{=} \lim_{n\to\infty} \frac{1-\cos(1/n)}{1/n} \quad ``0/0"$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\sin(1/n)\cdot -1/n^2}{-1/n^2}$$

$$= \sin(0) = 0 \;//$$

So sequence <u>converges</u> to 0.

(b) $a_n = \dfrac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}} \nearrow ``\infty - \infty"$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}} \cdot \frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{\sqrt{n^2-1}+\sqrt{n^2+n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{(n^2-1)-(n^2+n)} = \lim_{n\to\infty} \frac{\sqrt{n^2(1-1/n^2)}+\sqrt{n^2(1+1/n)}}{-1-n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(\sqrt{1-1/n^2}+\sqrt{1+1/n}\right)}{\cancel{n}\left(-1/n-1\right)}$$

$$= \frac{\sqrt{1}+\sqrt{1}}{-1} = \underline{\underline{-2}}$$

So sequence converges to -2.

5. $$(x^3+2x)\frac{dy}{dx} = y+1$$

$$\frac{1}{y+1}\cdot\frac{dy}{dx} = \frac{1}{x^3+2x} \Rightarrow \int \frac{1}{y+1}\,dy = \int \frac{1}{x^3+2x}\,dx$$

Now note, $$\underbrace{\int \frac{1}{y+1}\,dy}_{\substack{u=y+1 \\ du=dy}} = \int \frac{1}{u}\,du = \ln|u| + C$$
$$= \ln|y+1| + C_1$$

and $$\int \frac{1}{x^3+2x}\,dx = \int \frac{1}{x(x^2+2)}\,dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+2}\,dx$$

So $$1 = A(x^2+2) + (Bx+C)x$$

if $x=0 \Rightarrow 1 = 2A \Rightarrow A = 1/2$

now $A = 1/2$ implies

$$1 - 1/2(x^2+2) = Bx^2 + Cx$$

$$-1/2\,x^2 = Bx^2 + Cx \Rightarrow B = -1/2,\ C = 0$$

So,
$$\int \frac{1}{x^3+2x^2}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\underbrace{\int \frac{x}{x^2+2}\,dx}_{\substack{u=x^2+2 \\ du=2x\,dx \Rightarrow \frac{du}{2}=x\,dx}}$$

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = 1/2\ln|x^2+2| + C$$

$$= \frac{1}{2}\ln|x| - \frac{1}{4}\ln|x^2+2| + C_2$$

So, $$\ln|y+1| = \frac{1}{2}\ln|x| - \frac{1}{4}\ln|x^2+2| + C$$