NAME: Solutions

APPM 1360.300—Quiz 3: Sequences and Series—07/02/09

Clearly print your name above. Show all work to receive full credit. Justify all answers!

1. (2 points each) Determine whether or not the *sequences* $a_{n}{}_{n=1}^{\infty}$ with a_n given below converge or diverge. Explain your reasoning, and show all steps.

(a) $a_n = (-2)^n$

Diverges.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-2)^n$, limit does not exist (Note: limit is not infinity)

(b) $a_n = \frac{n}{e^n}$

$\lim_{n\to\infty} \frac{n}{e^n} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{e^n} = 0 < \infty$, therefore converges.

(Limit yields indeterminate form $\frac{\infty}{\infty}$)

(c) $a_n = \ln(n-1) - ln(3n+2)$, for $n \geq 2$

(Dominance of powers)

$$\lim_{n\to\infty}\left[\ln(n-1) - \ln(3n+2)\right] = \lim_{n\to\infty} \ln\left[\frac{n-1}{3n+2}\right] = \ln\left[\lim_{n\to\infty} \frac{n-1}{3n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}\right]$$

$$= \ln\left[\lim_{n\to\infty}\left(\frac{1-\frac{1}{n}}{3+\frac{2}{n}}\right)\right] = \ln\left[\frac{1}{3}\right],$$ therefore converges.

Quotient rule for limits

(d) $a_n = 2\cos(n\pi)$

$= 2\cdot(-1)^n = (-2)^n$

(Same sequence as #1d)

Therefore, the sequence diverges. Limit does not exist.

(e) a_n satisfies $0 < a_n < a_{n+1} < 1$ for all n

Sequence is positive, increasing, and bounded, therefore it converges.

(Non-decreasing sequence theorem)(pg. 619)

2. (5 points each) Determine whether or not the *series* given below converge or diverge. Find the value of the sum, if possible. Explain your reasoning, and show all steps.

(a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

(Use LCT) with $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{n}{n^2+1}\right)}{(1/n)} = \lim_{n\to\infty} \frac{n^2}{n^2+1} \frac{(1/n^2)}{(1/n^2)}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n^2} = 1 \text{ (finite, and nonzero)}$$

Therefore, because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by LCT, $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ also diverges.

(b) $\sum_{n=2}^{\infty} \frac{1}{n[\ln(n)]^2}$ (Note typo: series begins at $n=2$)

(Use Integral Test)

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{a\to\infty} \int_2^{a} \frac{dx}{x(\ln x)^2},$$ let $u = \ln x$, $du = \frac{dx}{x}$

$x = 2 \Rightarrow u = \ln 2$, $x = a \Rightarrow u = \ln a$

$$= \lim_{a\to\infty} \int_{\ln 2}^{\ln a} \frac{du}{u^2} = \lim_{a\to\infty} \left[-u^{-1}\right]_{\ln 2}^{\ln a} = \lim_{a\to\infty} \left[\frac{-1}{\ln a} + \frac{1}{\ln 2}\right] = \lim_{a\to\infty}\left[\frac{-1}{\ln a}\right] + \frac{1}{\ln 2}$$

$$= 0 + \frac{1}{\ln 2} < \infty$$

$\Rightarrow$ Convergence by Integral Test.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$

(Use AST), $u_n = \frac{1}{n+1}$

I.) $u_{n+1} < u_n$

II.) $u_n > 0$

III.) $\lim_{n\to\infty} u_n = 0$

All three conditions satisfied, therefore series converges.

(d) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(Use Ratio)

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| = \lim_{n\to\infty} \left|\frac{\cancel{(n+1)}\cdot \cancel{n!}}{\cancel{(n+1)}(n+1)^n} \cdot \frac{n^n}{\cancel{n!}}\right|$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = 1^{\infty}$$ (indeterminate form) $\Rightarrow$ Consider $\lim_{n\to\infty} \ln\left[\left(\frac{n}{n+1}\right)^n\right] = \ln L$

$$\lim_{n\to\infty} \ln\left[\left(\frac{n}{n+1}\right)^n\right] = \lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) \overset{L'H}{=} \lim_{n\to\infty} \frac{\left(\frac{n+1}{n}\right)\cdot\left(\frac{1}{(n+1)^2}\right)}{\left(\frac{-1}{n^2}\right)} = \lim_{n\to\infty}\left(-\frac{n}{n+1}\right) = -1$$

(Ind. Form $\frac{0}{0}$)

$\Rightarrow L = e^{-1} < 1$ $\Rightarrow$ convergence by ratio test