APPM 1360 – Summer 09 – Review #3 Solution.c

1. (a) R.O.C $= 1/2$, I.O.C $= (-7/2, -5/2]$

(b) abs. convergent on $(-7/2, -5/2)$

(c) cond conv @ $x = -5/2$

(d) divergent on $(-\infty, -7/2] \cup (-5/2, +\infty)$

(e) $$f'(x) = \sum_{n=1}^{\infty} \frac{(-2)^n n (x+3)^{n-1}}{\sqrt{n}}$$

$$\int f(x) = C + \sum_{n=1}^{\infty} \frac{(-2)^n (x+3)^{n+1}}{\sqrt{n}\,(n+1)}$$

2. Note $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

(a) Note $$\ln(1+x) = \int \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

(b) $$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}$$

(c) Using Alt. Series Est. Theorem:

$$\left| \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) \right| \le \frac{|x|^4}{4} \underbrace{\le}_{|x| < 0.1} \frac{(0.1)^4}{4}$$

so error $\le \frac{(0.1)^4}{4}$ using Alt. Series Est. Thm.

[redacted]

2 (d) By Alt. Series Est. Thm $\ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)$ has same sign as $\frac{-x^4}{4}$.

Note $|x| < 0.1 \Rightarrow -0.1 < x < 0.1$

[redacted]

and for $-0.1 < x < 0.1$ we have $\frac{-x^4}{4} < 0 \Rightarrow$ <u>overestimate</u>.

(3) Taylor Series of $f(x) = \sin(x)$ at $x = \pi/2$: $\sum_{n=0}^{\infty} \frac{(-1)^n (x - \pi/2)^{2n}}{(2n)!}$

(4) (a) $P_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2} x^2$

$= 1 + 3x + \frac{6}{2} x^2 = 1 + 3x + 3x^2$

$f(x) = (1+x)^3 \big|_{x=0} = 1$

$f'(x) = 3(1+x)^2 \big|_{x=0} = 3$

$f''(x) = 6(1+x) \big|_{x=0} = 6$

$f'''(x) = 6$

So, $(1+x)^3 \approx P_2(x) = 1 + 3x + 3x^2$

(b) $(1+x)^3 \approx P_2(x) \Rightarrow |(1+x)^3 - P_2(x)| \leq |R_2(x)|$ [redacted]

where $R_2(x) = \frac{f^{(3)}(c)}{3!} x^3 \underset{\uparrow}{=} \frac{6}{6} x^3 = x^3$, continued on next page

$f^{(3)}(c) = 6$

4 (b) continued...

Note $|R_2(x)| = |x|^3$

we want $|R_2(x)| < \frac{1}{1000}$

i.e. $|x|^3 < \frac{1}{1000} \Rightarrow |x| < \frac{1}{10} \Rightarrow |x| < 0.1$

So for $-0.1 < x < 0.1$ we will have error $< 1/1000$.

5. (a) Abs. convergent by Ratio Test

(b) $\sum_{k=1} (-1)^k \frac{\ln(k)}{\sqrt{k}}$ - conditionally convergent

Note $\sum_{k=3} \frac{\ln(k)}{\sqrt{k}} > \sum_{k=3} \frac{1}{\sqrt{k}}$

so $\sum_{k=1} \frac{\ln(k)}{\sqrt{k}}$ diverges by PCT w/ $\sum \frac{1}{\sqrt{k}}$

and $\sum_{k=1}^{\infty} \frac{(-1)\ln(k)}{\sqrt{k}}$ converges by A.S.T. (you show ① $u_{n+1} \leq u_n$ ② $\lim_{n\to\infty} u_n = 0$)

Cont. on next page...

5 ⓒ $\displaystyle\sum_{n=2}^{\infty}\left(\frac{2}{n^2-1}+\frac{4}{5^n}\right)=\frac{3}{2}+1/5={}^{17}/_{10}$, Abs Convergent

Note $\displaystyle\frac{2}{n^2-1}=\underbrace{\frac{2}{(n-1)(n+1)}=\frac{1}{n-1}-\frac{1}{n+1}}_{\text{by Partial fractions}}$

So, $\displaystyle\sum_{n=2}^{\infty}\frac{2}{n^2-1}=\sum_{n=2}^{\infty}\frac{1}{n-1}-\frac{1}{n+1}=\left(1-\tfrac{1}{3}\right)+\left(\tfrac{1}{2}-\tfrac{1}{4}\right)+\left(\tfrac{1}{3}-\tfrac{1}{5}\right)+\left(\tfrac{1}{4}-\tfrac{1}{6}\right)+\dots$

$$=1+1/2-\lim_{n\to\infty}\frac{1}{n+1}=1+1/2={}^{3}/_{2}$$

and, $\displaystyle\sum_{n=2}^{\infty}\frac{4}{5^n}=\sum_{n=2}^{\infty}\frac{4}{25}\left(\frac{1}{5}\right)^{n-2}=\frac{4/25}{1-1/5}=\frac{4/25}{4/5}=1/5$

So $\displaystyle\sum_{n=2}^{\infty}\frac{2}{n^2-1}+\frac{4}{5^n}=\frac{3}{2}+1/5={}^{17}/_{10}$

ⓓ Abs. Convergent by Integral Test.

ⓔ Abs. Convergent by DCT w/ $\sum\frac{2}{n^5}$

Note $\displaystyle 0<\frac{1+\cos^2(x)}{1+n^5}<\frac{2}{1+n^5}<\frac{2}{n^5}$

So $\displaystyle 0<\sum_{n=1}^{\infty}\frac{1+\cos^2(x)}{1+n^5}<\sum_{n=1}^{\infty}\frac{2}{n^5}$. ← convergent p-series

ⓕ Diverges by LCT w/ $\sum 1/n$.