APPM 1360 Test 1 solns
FA2009

1. $$SA = \int_1^2 2\pi f(y)\sqrt{1+f'(y)^2}\,dy$$

$$f'(y) = \frac{1}{2}\cdot\frac{4-2y}{\sqrt{4y-y^2}} = \frac{2-y}{\sqrt{4y-y^2}}$$

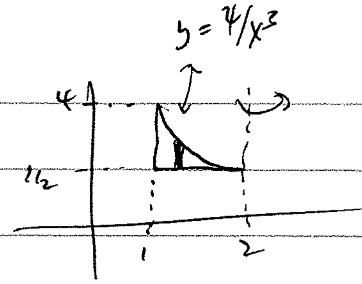
so $$1+f'(y)^2 = 1+\left(\frac{2-y}{\sqrt{4y-y^2}}\right)^2 = 1+\frac{4-4y+y^2}{4y-y^2}$$

$$= \frac{4y-y^2}{4y-y^2}+\frac{4-4y+y^2}{4y-y^2} = \frac{4}{4y-y^2}$$

so $$SA = \int_1^2 2\pi\sqrt{4y-y^2}\cdot\sqrt{\frac{4}{4y-y^2}}\,dy$$

$$= 2\pi\int_1^2 2\,dy = 4\pi y\Big|_1^2 = \underline{\underline{4\pi}}$$

2 (a)



Shell Method: $r = 2 - x$

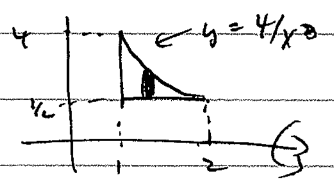
$h = \frac{4}{x^3} - 1/2$

so

$$V = \int_1^2 2\pi(2-x)\left(\frac{4}{x^3} - 1/2\right)\,dx$$

$$= 2\pi\left[\frac{-4}{x^3} - x + \frac{4}{x} + \frac{1}{4}x^2\right]\Big|_1^2 = \underline{\underline{\frac{9\pi}{2}}}$$

(b)



Disk Method / Washer Method:

$R = 4/x^3$

$r = 1/2$

so $$V = \int_1^2 \pi\left[\left(4/x^3\right)^2 - \left(1/2\right)^2\right]dx$$

$$= \int_1^2 \pi\left[\frac{16}{x^6} - 1/4\right]dx$$

3 (a) $A = \int_{-1}^{1} [3 - x^2 - 2x^2] \, dx = \int_{-1}^{1} (3 - 3x^2) \, dx$

$= 3\left(x - x^3/_3\right)\Big|_{-1}^{1} = 4$

(b) $\int dm = \text{Mass} = \int_{-1}^{1} \delta (3 - 3x^2) \, dx = 4\delta$ from part (a)

$\bar{x} = 0$ by symmetry

Here $\tilde{y} = \dfrac{2x^2 + (3 - x^2)}{2} = \dfrac{x^2 + 3}{2}$

so,

$M_x = \int \tilde{y} \, dm = \int_{-1}^{1} \left(\dfrac{x^2 + 3}{2}\right) \delta \cdot (3 - 3x^2) \, dx$

$= \dfrac{3\delta}{2} \int_{-1}^{1} (x^2 + 3)(1 - x^2) \, dx$

$= \dfrac{3\delta}{2} \int_{-1}^{1} \left(3 - 2x^2 - x^4\right) dx$

$= \dfrac{3\delta}{2} \left(3x - \dfrac{2x^3}{3} - \dfrac{x^5}{5}\right)\Big|_{-1}^{1} = \dfrac{32}{5} \delta$

so $\bar{y} = \dfrac{32\delta/5}{4\delta} = 8/5$

so $(\bar{x}, \bar{y}) = (0, 8/5)$.

3 (c) $L = \int_{-1}^{1} \sqrt{1+4x^2}\, dx + \int_{-1}^{1} \sqrt{1+16x^2}\, dx$

4 (a) $\sinh x \cdot y' + 3\cosh x \cdot y = \cosh x \sinh x$

$$y' + 3\coth(x) \cdot y = \cosh(x)$$

so $v(x) = e^{\int 3\coth(x)\, dx} = e^{3\ln|\sinh(x)|}$

$$= \sinh^3(x)$$

so $y = \frac{1}{v(x)} \left[\int Q(x)\, v(x) \right]$

$$= \frac{1}{\sinh^3(x)} \int \sinh^3(x) \cosh(x)\, dx$$

let $u = \sinh(x)$ $\leftarrow$ $= \frac{1}{\sinh^3(x)} \left[\frac{(\sinh(x))^4}{4} + C \right]$

so $y = \frac{\sinh(x)}{4} + \frac{C}{\sinh^3(x)}$

4(b) $\sec(x)\, y' = e^{\sin(x) - y}$

$$\sec(x)\, y' = \frac{e^{\sin(x)}}{e^y}$$

so $e^y y' = \cos(x) e^{\sin(x)}$

so $$\int e^y\, dy = \underbrace{\int \cos(x)\, e^{\sin(x)}\, dx}_{\text{let } u = \sin(x)}$$

$$e^y = e^{\sin(x)} + C$$

so $y = \ln\left(e^{\sin(x)} + C\right)$.

5 (a) $y = \sin^{-1}(x) - x\,\operatorname{sech}^{-1}(2x)$

so $$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \operatorname{sech}^{-1}(x) + x \cdot \left(\frac{-2}{2x\sqrt{1-4x^2}}\right)$$

$$= \frac{1}{\sqrt{1-x^2}} - \operatorname{sech}^{-1}(x) - \frac{1}{\sqrt{1-4x^2}}.$$

(b) $$\int_{\pi}^{\sqrt{e}} \ln\left(\cosh(x) + \sinh(x)\right) dx = \int_{\pi}^{\sqrt{e}} \ln\left(e^x\right) dx$$

$$= \int_{\pi}^{\sqrt{e}} x\, dx = \left.\frac{x^2}{2}\right|_{\pi}^{\sqrt{e}} = \frac{e - \pi^2}{2}.$$

(Recall $\cosh(x) = \frac{e^x + e^{-x}}{2}$, $\sinh(x) = \frac{e^x - e^{-x}}{2}$, so $\cosh(x) + \sinh(x) = e^x$)