1. Shell Method: $\Delta V = 2\pi r h \Delta x$ where $r = x,\ h = \sin(x)$, so $V = \int_0^\pi 2\pi x \sin(x) dx$ and using integration by parts we have,

$$\int x\sin(x)dx \underbrace{=}_{u=x,\ dv=\sin(x)dx} -x\cos(x) + \int \cos(x)dx = -x\cos(x) + \sin(x)$$

so $V = 2\pi\big(-x\cos(x) + \sin(x)\Big|_0^\pi\big) = 2\pi^2$.

2. Note $(x^2+x)\dfrac{dy}{dx} = (y-9)^2$ implies $\displaystyle\int \frac{1}{(y-9)^2}dy = \int \frac{1}{x^2+x}dx$ where

$$\int \frac{1}{(y-9)^2}dy \underbrace{=}_{u=y-9,\ du=dy} \int u^{-2}du = -u^{-1} + C_1 = -\frac{1}{y-9} + C_1$$

and using partial fractions, we have

$$\int \frac{dx}{x^2+x} = \int \frac{dx}{x(x+1)} \underbrace{=}_{\frac{1}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}} = \int \frac{1}{x} + \frac{-1}{x+1}dx = \ln|x| - \ln|x+1| + C_2$$

so we have,

$$\frac{-1}{y-9} = \ln\left|\frac{x}{x+1}\right| + C.$$

3.Using trigonometric substitution,

$$\int \frac{4x^2}{(1-x^2)^{3/2}}dx \underbrace{=}_{\cos(\theta)=\sqrt{1-x^2},\ \sin(\theta)=x} \int \frac{4\sin^2(\theta)}{\cos^3(\theta)}\cos(\theta)d\theta = 4\int \tan^2(\theta)d\theta = 4\int(\sec^2(\theta)-1)d\theta$$

now note that $4\displaystyle\int(\sec^2(\theta)-1)d\theta = 4(\tan(\theta)-\theta) + C$, so

$$\int \frac{4x^2}{(1-x^2)^{3/2}}dx = \frac{4x}{\sqrt{1-x^2}} - 4\sin^{-1}(x) + C.$$

4. (a) Note that $-1 \le (-1)^n \le 1$ implies $-n \le n(-1)^n \le n$, for $n > 0$ and so $1-n \le 1+n(-1)^n \le 1+n$, thus

$$\frac{1-n}{n!} \le \frac{1+n(-1)^n}{n!} \le \frac{1+n}{n!}$$

and $\displaystyle\lim_{n\to\infty}\frac{1-n}{n!} = \lim_{n\to\infty}\frac{1}{n!} - \frac{1}{(n-1)!} = 0-0 = 0$, similarly $\displaystyle\lim_{n\to\infty}\frac{1+n}{n!} = 0$ and so by the Squeeze Theorem we have that

$$\lim_{n\to\infty}\frac{1+n(-1)^n}{n!} = 0$$

and so a_n converges to 0 as $n\to\infty$.

4. (b) Note that

$$\lim_{n\to\infty}\ln(n)^{1/n} = \lim_{n\to\infty}e^{\ln(\ln(n)^{1/n})} = \lim_{n\to\infty}e^{\ln(\ln(n))/n} \overset{L'H}{=} \lim_{n\to\infty}e^{1/n\ln(n)} = e^0 = 1$$

so $\sqrt[n]{\ln(n)} \to 1$ as $n\to\infty$.

4. (c) Note that $f(x) = \dfrac{1}{x(\ln(x))^3}$ is *positive, continuous* and *decreasing* and note that

$$\int \frac{1}{x(\ln(x)^3)}dx \underbrace{=}_{u=\ln(x),\ du=dx/x} \int u^{-3}du = \frac{u^{-2}}{-2} + C = \frac{-1}{2(\ln(x))^2} + C$$

so,

$$\int_{21}^{\infty} \frac{1}{x(\ln(x)^3)} dx = \lim_{b\to\infty} \left. \frac{-1}{2(\ln(x))^2} \right|_{21}^{b} = \lim_{b\to\infty} \frac{-1}{2(\ln(b))^2} + \frac{1}{2(\ln(21))^2} = \frac{1}{2(\ln(21))^2}$$

so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^3}$ converges by the Integral Test.

4. (d) Note that $\sqrt{t} \geq 1$ for $t \geq 1$ and so $e^t\sqrt{t} \geq e^t$ which implies $\dfrac{1}{e^t} \geq \dfrac{1}{\sqrt{t}e^t}$ for $t \geq 1$ and so

$$0 \leq \int_{88}^{\infty} \frac{1}{\sqrt{t}e^t} dt \leq \int_{88}^{\infty} \frac{1}{e^t} dt$$

and note

$$\int_{88}^{\infty} \frac{1}{e^t} dt = \lim_{b\to\infty} \left. \frac{-1}{e^t} \right|_{88}^{b} = \lim_{b\to\infty} \frac{-1}{e^b} + \frac{1}{e^{88}} = \frac{1}{e^{88}}$$

so $\displaystyle\int_{88}^{\infty} \frac{1}{\sqrt{t}e^t} dt$ converges by Limit Comparison Test with $\displaystyle\int_{88}^{\infty} \frac{1}{e^t} dt$.

5. (a) Note that

$$\begin{aligned} \sum_{k=1}^{\infty} \Big[\tan^{-1}(k-1) - \tan^{-1}(k) \Big] &= \Big[\tan^{-1}(0) - \tan^{-1}(1) \Big] + \Big[\tan^{-1}(1) - \tan^{-1}(2) \Big] + \Big[\tan^{-1}(2) - \cdots \\ &= \tan^{-1}(0) - \lim_{k\to\infty} \tan^{-1}(k) = 0 - \frac{\pi}{2} = -\frac{\pi}{2} \end{aligned}$$

so this is a convergent Telescoping series converging to $-\pi/2$.

5. (b) Note that $2 < e$ implies $\ln(2) < \ln(e)$, *i.e.* $\ln(2) < 1$, and since we are dealing with a Geometric Series, we have,

$$\sum_{n=0}^{\infty} (\ln(2))^n = \frac{1}{1 - \ln(2)}$$

and $4 > e$ implies $\ln(4) > 1$ so $1 > \frac{1}{\ln(4)}$ and thus

$$\sum_{n=0}^{\infty} \left(\frac{1}{\ln(4)} \right)^n = \frac{1}{1 - \frac{1}{\ln(4)}} = \frac{\ln(4)}{\ln(4) - 1}$$

and so,

$$\sum_{n=0}^{\infty} \left[(\ln 2)^n + \frac{1}{(\ln 4)^n} \right] = \frac{1}{1 - \ln(2)} + \frac{\ln(4)}{\ln(4) - 1}$$

thus this is a covergent Geometric series.

5. (c) Note that $\displaystyle\lim_{j\to\infty} \frac{j}{2j+1} = \frac{1}{2} \neq 0$ so the series diverges by the nth term Divergence Test.