1.(a) We know $\int_9^\infty e^x\,dx = \lim_{b\to\infty}\int_9^b e^x\,dx = \lim_{b\to\infty}(e^b - e^9)$ diverges.

Also, $0 < e^x < e^{x^2}$ for all $x \geq 9$ so $\int_9^\infty e^x\,dx < \int_9^\infty e^{x^2}\,dx$

Since $\int_9^\infty e^x\,dx$ diverges, $\int_9^\infty e^{x^2}\,dx$ diverges by comparison test.

(b) $a_n = n(1-\cos(\frac{1}{n}))$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\left(1-\cos\tfrac{1}{n}\right) = \lim_{n\to\infty}\frac{1-\cos(\frac{1}{n})}{\frac{1}{n}}$$

$$\overset{\text{l'Hopital}}{=} \lim_{n\to\infty}\frac{\sin(\frac{1}{n})(-\frac{1}{n^2})}{-\frac{1}{n^2}} = \lim_{n\to\infty}\sin\left(\tfrac{1}{n}\right) = \sin 0 = 0$$

$\therefore$ The sequence $\{a_n\}$ converges to 0.

(c) $\sum_{n=1}^\infty (-1)^n 2^{n+1} x^n = 2\sum_{n=1}^\infty (-2x)^n$ is a geometric series

with $r = -2x$ and $a = -4x$.

It converges whenever $|r| < 1 \Rightarrow |-2x| < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

For $-\frac{1}{2} < x < \frac{1}{2}$, $2\sum_{n=1}^\infty(-2x)^n = \dfrac{-4x}{1+2x}$ $\Leftarrow$ sum of series

(d) Let $a_n = (1-\frac{1}{n^2})^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1-\tfrac{1}{n^2}\right)^n = \lim_{n\to\infty} e^{n\ln(1-\frac{1}{n^2})} = \lim_{n\to\infty} e^{\left(\frac{\ln(1-\frac{1}{n^2})}{\frac{1}{n}}\right)}$$

$$\overset{\text{l'Hopital}}{=} \lim_{n\to\infty} e^{\left(\frac{1}{1-\frac{1}{n^2}}\cdot\frac{2}{n^3}\right)/-\frac{1}{n^2}} = \lim_{n\to\infty} e^{\frac{-2}{(1-\frac{1}{n^2})n}} = e^0 = 1$$

Since $\lim_{n\to\infty} a_n = e^0 = 1$ the series $\sum_{n=1}^\infty (1-\frac{1}{n^2})^n$ diverges.

2(a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\dfrac{-8}{(4n-3)(4n+1)} = 0$. The sequence $\{a_n\}$ converges to 0.

(b) $\dfrac{-8}{(4n-3)(4n+1)} = \dfrac{A}{4n-3} + \dfrac{B}{4n+1} \Rightarrow -8 = A(4n+1) + B(4n-3)$

$\Rightarrow 4A + 4B = 0 \Rightarrow A = -B$

and $A - 3B = -8 \Rightarrow B = 2$

$$\therefore S_n = \sum_{k=1}^n \frac{-8}{(4k-3)(4k+1)} = \sum_{k=1}^n\left[\frac{-2}{4n-3} + \frac{2}{4n+1}\right]$$

$$= \left(-2 + \tfrac{2}{5}\right) + \left(-\tfrac{2}{5} + \tfrac{2}{9}\right) + \left(-\tfrac{2}{9} + \tfrac{2}{13}\right) + \cdots + \left(\tfrac{-2}{4n-3} + \tfrac{2}{4n+1}\right)$$

$\boxed{S_n = -2 + \frac{2}{4n+1}}$ and $\boxed{S_{10} = -2 + \frac{2}{41} = -\frac{79}{41}}$

2 (c) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(-2 + \frac{2}{4n+1}\right) = -2$

Yes, the sequence of partial sums $\{S_n\}$ converges to -2.

(d) $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{-8}{(4k-3)(4k+1)} = \lim_{n\to\infty} S_n = -2$

$\therefore$ The series $\sum_{k=1}^{\infty} a_k$ converges to -2.

3(a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$. Let $f(x) = \frac{1}{x(\ln x)^p}$.

u-sub: $u = \ln x$

For $p \neq 1$: $\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^p}\,dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^p}\,du$

$= \lim_{b\to\infty} \left[\frac{u^{-p+1}}{1-p}\right]_{\ln 2}^{\ln b} = \lim_{b\to\infty} \left[\frac{(\ln b)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}\right]$

We note that $\lim_{b\to\infty} (\ln b)^{1-p} = \begin{cases} 0 & \text{if } 1-p<0 \\ \text{diverges} & \text{if } 1-p>0 \end{cases}$

Also, if $p=1$: $\int_2^{\infty} \frac{1}{x\ln x}\,dx$ also diverges

$\therefore$ By the integral test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$.

(b) $\int_1^2 \frac{1}{x(\ln x)^p}\,dx = \lim_{a\to 1^+} \int_a^2 \frac{1}{x(\ln x)^p}\,dx$ ← note: same integral as part a) but limits of integration are different. (Integrand is undefined when $x=1$.)

assume $p\neq 1$, $u=\ln x$, $du = \frac{1}{x}dx$

$= \lim_{a\to 1^+} \int_{\ln a}^{\ln 2} \frac{1}{u^p}\,du$

$= \lim_{a\to 1^+} \left[\frac{(\ln 2)^{1-p}}{1-p} - \frac{(\ln a)^{1-p}}{1-p}\right]$

Since $\lim_{a\to 1^+} \ln a = 0$ we will have convergence only if $1-p>0 \Rightarrow p<1$. (Diverges if $p\geq 1$.)

Answer: $\int_1^2 \frac{1}{x(\ln x)^p}\,dx$ is an improper integral that converges if $p<1$.

4(a) $\int e^{3t}\sin 2t\,dt$ Let $u = e^{3t}$ $dv = \sin 2t\,dt$

$du = 3e^{3t}\,dt$ $v = -\frac{1}{2}\cos 2t$

$= e^{3t}\left(-\frac{1}{2}\cos 2t\right) + \frac{3}{2}\int e^{3t}\cos 2t\,dt$ ← do integration by parts a second time

$u = e^{3t}$ $dv = \cos 2t\,dt$

$du = 3e^{3t}\,dt$ $v = \frac{1}{2}\sin 2t$

$= -\frac{1}{2}e^{3t}\cos 2t + \frac{3}{2}\left[\frac{1}{2}e^{3t}\sin 2t - \frac{3}{2}\int e^{3t}\sin 2t\,dt\right]$

$= -\frac{1}{2}e^{3t}\cos 2t + \frac{3}{4}e^{3t}\sin 2t - \frac{9}{4}\int e^{3t}\sin 2t\,dt$

$\therefore \frac{13}{4}\int e^{3t}\sin 2t\,dt = -\frac{1}{2}e^{3t}\cos 2t + \frac{3}{4}e^{3t}\sin 2t$ (multiply both sides by $\frac{4}{13}$)

$\Rightarrow \int e^{3t}\sin 2t\,dt = -\frac{2}{13}e^{3t}\cos 2t + \frac{3}{13}e^{3t}\sin 2t + C$

(b) $\int_0^1 \frac{dx}{(4-x^2)^{3/2}}$ Let $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

when $x = 0$, $\theta = 0$

$x = 1$ $\theta = \frac{\pi}{6}$

$= \int_0^{\pi/6} \frac{2\cos\theta\,d\theta}{(4-4\sin^2\theta)^{3/2}}$

$= \int_0^{\pi/6} \frac{2\cos\theta}{8(1-\sin^2\theta)^{3/2}}\,d\theta = \frac{1}{4}\int_0^{\pi/6}\frac{1}{\cos^2\theta}\,d\theta = \frac{1}{4}\int_0^{\pi/6}\sec^2\theta\,d\theta$

$= \frac{1}{4}\tan\theta\,\Big|_0^{\pi/6} = \frac{1}{4}\left[\tan\left(\frac{\pi}{6}\right) - \tan 0\right] = \frac{1}{4}\cdot\frac{1}{\sqrt{3}} = \boxed{\frac{1}{4\sqrt{3}}}$

5.(a) False (For a sequence of positive numbers, bounded from above to converge, the seq. must be nondecreasing. Counterexample: 1, 2, 1, 2, 1, 2, ...)

(b) False (must have $\sum a_n$ and $\sum b_n$ convergent, See Thm 7, p.636)

(c) True (reindexing)

(d) True (comparison test)

(e) False (Surface area $S = \int_1^\infty \frac{2\pi}{x}\sqrt{1+\frac{1}{x^2}}\,dx$ improper integral diverges, compare to $\int_1^\infty \frac{2\pi}{x}\,dx$)

(f) True (Volume $V = \int_1^\infty \frac{\pi}{x^2}\,dx$ which converges.)