APPM 1360 - Exam 3 - Solutions

1 (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2+1}}$. Since $n \le \sqrt{n^2+1}$ we have $\frac{1}{\sqrt{n^2+1}} \le \frac{1}{n}$

$$\therefore \left| \frac{(-1)^n}{n\sqrt{n^2+1}} \right| = \frac{1}{n\sqrt{n^2+1}} \le \frac{1}{n^2}$$

We know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges absolutely (p-series, $p=2$)

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2+1}}$ converges absolutely by the direct comparison test.

(b) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ Use ratio test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}$

$$= \lim_{n\to\infty} \frac{3}{n+1} = 0 = \rho$$

Since $\rho = 0 < 1$ we know the series converges absolutely.

2 $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^{2n}\sqrt{n^2+1}}$

To find interval of convergence, use ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{3^{2(n+1)}\sqrt{(n+1)^2+1}} \cdot \frac{3^{2n}\sqrt{n^2+1}}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| (x-2) \frac{3^{2n}}{3^{2n+2}} \cdot \frac{\sqrt{n^2+1}}{\sqrt{(n+1)^2+1}} \right|$$

$$= \frac{|x-2|}{9} \lim_{n\to\infty} \sqrt{\frac{n^2+1}{(n+1)^2+1}} = \frac{|x-2|}{9}$$

$$\frac{|x-2|}{9} < 1 \Rightarrow -9 < x-2 < 9 \Rightarrow -7 < x < 11$$

Check endpts:

$x=-7$: $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n}}{3^{2n}\sqrt{n^2+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$

Use AST:

(1) $a_n = \frac{1}{\sqrt{n^2+1}} > 0$

(2) $a_{n+1} = \frac{1}{\sqrt{(n+1)^2+1}} < \frac{1}{\sqrt{n^2+1}} = a_n$

(3) $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$.

$\therefore$ Converges conditionally by AST

(Note: Does not converge absolutely by limit comparison test with $1/n$. See equivalent work at $x=11$.)

$x=11$: $\sum_{n=1}^{\infty} \frac{3^{2n}}{3^{2n}\sqrt{n^2+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

Since $\lim_{n\to\infty} \frac{1/n}{1/\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n} = \lim_{n\to\infty} \sqrt{1 - 1/n^2} = 1$

we know $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges by limit comparison test.

continued

$\therefore$ 2(a) interval of convergence: $-7 \le x < 11$

radius of convergence is 9

(b) Converges absolutely: $-7 < x < 11$

(c) converges conditionally: $x = -7$

(d) diverges for $x < -7$, $x \ge 11$.

3 (a) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$

$$\sin(2x)\cos x = \left(\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right)$$

$$= \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= 2x - x^3\left(\frac{8}{3!} + \frac{2}{2!}\right) + x^5\left(\frac{2^5}{5!} + \frac{2^3}{2!3!} + \frac{2}{4!}\right) - \cdots$$

$$= \underbrace{2x - \frac{7}{3}x^3 + \frac{122}{5!}x^5}_{\text{first 3 nonzero terms}} + \cdots$$

(b) $f(x) = \ln(\sin x)$ | $f(\frac{\pi}{4}) = \ln\left(\frac{\sqrt{2}}{2}\right)$

$f'(x) = \frac{\cos x}{\sin x}$ | $f'(\frac{\pi}{4}) = 1$

$f''(x) = \frac{-1}{\sin^2 x}$ | $f''(\frac{\pi}{4}) = \frac{-1}{\left(\frac{\sqrt{2}}{2}\right)^2} = -2$

$\therefore P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

$= \ln\left(\frac{\sqrt{2}}{2}\right) + \left(x - \frac{\pi}{4}\right) - \left(x - \frac{\pi}{4}\right)^2$

4 (a) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $-1 < x < 1$

(b) $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$, $-1 < x < 1$

(c) $\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2}$, $-1 < x < 1$

(d) $\frac{2x^2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^n$, $-1 < x < 1$

(e) Substitute $x = 1/3$ in both sides: $\frac{2\left(\frac{1}{3}\right)^2}{\left(1-\frac{1}{3}\right)^3} = \frac{\frac{2}{9}}{\frac{8}{27}} = \boxed{\frac{3}{4} = \sum_{n=2}^{\infty} n(n-1)\left(\frac{1}{3}\right)^n}$

5. $\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$

(a) $\frac{\sin t}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$

(b) $\int_0^x \frac{\sin t}{t}\,dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!}\Big|_0^x$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$

(c) $Si(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$

Estimate $Si(1) \approx 1 - \frac{1}{3\cdot 3!} = 1 - \frac{1}{18} = \boxed{\frac{17}{18}}$

(d) $|\text{error}| \le |\,1^{\text{st}}$ unused term$\,|$ since $Si(1)$ is an alternating series.

$= \frac{1}{5\cdot 5!} = \frac{1}{600}$

6. (a) False, focus of the parabola is $(\frac{5}{2}, 0)$

(b) $x^2+4x+y^2=12$

$x^2+4x+4+y^2=16$ False

$(x+2)^2+y^2=16 \Rightarrow$ circle, center at $(-2,0)$

(c) $5x^2-4y^2=0$

$\frac{x^2}{4} - \frac{y^2}{5} = 1$ False, hyperbola, asymptotes at $y = \pm\frac{\sqrt{5}}{2}x$