1. (30 pts) Determine if the integrals converge or diverge. Each part is worth 10 pts.

(a) $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx$ (b) $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ (c) $\displaystyle\int_1^\infty e^{x^2}\,dx$

Solution:

(a) Writing the integral as a limit, we have

$$\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{e^x}{e^{2x}+1}\,dx \underbrace{=}_{u=e^x} \lim_{t\to\infty} \tan^{-1}(e^t) - \tan^{-1}(1) = \pi/2 - \pi/4 = \pi/4$$

so the integral converges to $\pi/4$.

(b) Note that

$$0 < \int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx < \int_1^\infty \frac{x}{\sqrt{x^6}}\,dx = \int_1^\infty \frac{1}{x^2}\,dx$$

and note that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, and so $\displaystyle\int_1^\infty \frac{xdx}{\sqrt{1+x^6}}$ converges by Direct Comparison Test.

(c) Note $x^2 \geq x$ for $x \geq 1$ and so

$$\int_1^\infty e^{x^2} \geq \int_1^\infty e^x > 0$$

and $\displaystyle\int_1^\infty e^x = \lim_{t\to\infty} e^t - e = \infty$ and so $\displaystyle\int_1^\infty e^{x^2}$ diverges by Direct Comparison Test.

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2. (16 pts) Find the antiderivative. Each part is worth 8 pts.

(a) $\displaystyle\int \sqrt{1-4x^2}\,dx$ (b) $\displaystyle\int \frac{\cos(x)\sin(x)}{\sin^2(x)+3\sin(x)+2}\,dx$

Solution:

(a)Using the substitution $x = \sin(\theta)/2$, we have

$$\begin{aligned}\int \sqrt{1-4x^2}\,dx &= \frac{1}{2}\int \cos^2(\theta)\,d\theta = \frac{1}{4}\int [1+\cos(2\theta)]\,d\theta \\ &= \frac{1}{4}\left[\theta + \frac{\sin(2\theta)}{2}\right] = \frac{1}{4}\left[\theta + \frac{2\sin(\theta)\cos(\theta)}{2}\right] = \frac{1}{4}\left[\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\right] + C\end{aligned}$$

(b) If we let $u = \sin(x)$ then $du = \cos(x)dx$ and using partial fractions yields

$$\int \frac{u}{u^2+3u+2}\,du = \int \frac{2}{u+2}\,du + \int \frac{-1}{u+1}\,du = 2\ln|\sin(x)+2| - \ln|\sin(x)+1| + C$$

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3. (24 pts) Evaluate the given integrals. Each part is worth 8 pts.

(a) $\displaystyle\int_1^2 xe^x\,dx$ (b) $\displaystyle\int_{-1}^1 \frac{e^x}{-1+e^x}\,dx$ (c) $\displaystyle\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx$

Solution:

(a) Using integration by parts with $u = x$ and $dv = e^x dx$ we get,

$$\int_1^2 xe^x \, dx = xe^x\Big|_1^2 - \int_1^2 e^x \, dx = e^x(x-1)\Big|_1^2 = e^2$$

(b) Note that the integral is undefined at $x = 0$, so

$$\begin{aligned}
\int_{-1}^{1} \frac{e^x}{-1+e^x} \, dx &= \int_{-1}^{0} \frac{e^x}{-1+e^x} \, dx + \int_{0}^{1} \frac{e^x}{-1+e^x} \, dx \\
&= \lim_{t\to 0^-} \int_{-1}^{t} \frac{e^x}{-1+e^x} \, dx + \lim_{t\to 0^+} \int_{t}^{1} \frac{e^x}{-1+e^x} \, dx \\
\{u = e^x - 1\} \to \quad &= \lim_{t\to 0^-} \ln|e^t - 1| - \ln|e^{-1} - 1| + \lim_{t\to 0^+} \ln|e^1 - 1| - \ln|e^t - 1| \\
&= \text{“}\infty - \infty\text{''}
\end{aligned}$$

so the integral diverges.

(c) Using the u-substitution $u = x^2 + 1$, then $du = 2xdx$ and so

$$\int_0^{\sqrt{3}} x\sqrt{x^2+1} \, dx = \frac{1}{2}\int_1^4 \sqrt{u} \, du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^4 = \frac{(4)^{3/2}}{3} - \frac{1}{3} = 7/3.$$

(Alternate method, trigonometric substitution:

Using the substitution $x = \tan(\theta)$, we get

$$\int x\sqrt{x^2+1} \, dx = \int \tan(\theta)\sec(\theta)\cdot\sec^2(\theta) \, d\theta \underbrace{=}_{u=\sec(\theta)} \int u^2 \, du = \frac{u^3}{3} = \frac{1}{3}(\sec(\theta))^3 = \frac{(\sqrt{x^2+1})^3}{3}$$

so,

$$\int_0^{\sqrt{3}} x\sqrt{x^2+1} \, dx = \frac{(x^2+1)^{3/2}}{3}\Big|_0^{\sqrt{3}} = 8/3 - 1/3 = 7/3.)$$

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4. (a) (9 pts) What is the area of one side of a thin metal plate that covers the region above the x-axis and below the curve $y = \ln(x)$, $e \le x \le e^2$.

   (b) (6 pts) Set-up but **do not solve** an integral (or integrals) to find the area of the region bounded by the curves $x^2 + y = 4$, $x + y = 2$ and the lines $x = -2$ and $x = 2$.

Solution:

(a) Here the area is given by $\int_e^{e^2} \ln(x) \, dx$, now using integration by parts with $u = \ln(x)$ and $dv = dx$ yields

$$\int_e^{e^2} \ln(x) \, dx = x\ln(x)\Big|_e^{e^2} - \int_e^{e^2} dx = x(\ln(x) - 1)\Big|_e^{e^2} = e^2$$

(b) Here the area is given by

$$A = \int_{-2}^{-1} \left[(2-x) - (4-x^2)\right] \, dx + \int_{-1}^{2} \left[(4-x^2) - (2-x)\right] \, dx$$

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5. (15 pts) **Always True** or **False**: Do not justify your answer, just answer either *Always True* or *False*, each part is worth 3 points:

(a) $\int \cos^2(x)\, dx = \frac{\cos^3(x)}{3} + C$

(b) If $f(x)$ is continuous, then $\int_{-\infty}^{\infty} f(x)\, dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)\, dx$

(c) Given the integral $\int \sqrt{x^2-1}\, dx$, one possible trigonometric substitution to use to solve the integral is $x = \tan(\theta)$.

(d) The partial fraction decomposition of $f(x) = \frac{1}{x^2(x^2+60)}$ is $\frac{1}{x^2(x^2+60)} = \frac{Ax+B}{x^2} + \frac{Cx+D}{x^2+60}$.

(e) Suppose, for functions $f(x)$ and $m(x)$, we have that $f(x) \geq m(x) > 0$ for all $x > 106$ then if $\int_{106}^{\infty} m(x)$ diverges then so does $\int_{106}^{\infty} f(x)$.

Solution:

(a) False (b) False (c) False (d) True (e) True