1. (18 pts) Answer the following questions, justify your answers.

   (a) (6 pts) Evaluate the limit using series: $\lim_{x\to 0} \dfrac{\cos(x^2)-1}{x^4}$

   (b) (6 pts) Find the first four terms of the Maclaurin series of $f(x) = \dfrac{x^2}{\sqrt{3+x}}$, simplify your answer.

   (c) (6 pts) Find the Maclaurin series of $f(x) = x^3 \sin(3x)$, give your answer in $\sum$-notation.

   Solution:

   (a) Since $\cos(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}$, we have $\cos(x^2) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{4n}}{(2n)!} = 1 - \dfrac{x^4}{2!} + \dfrac{x^8}{4!} - \dfrac{x^{12}}{6!} + \cdots$, so

$$\lim_{x\to 0} \frac{\cos(x^2)-1}{x^4} = \lim_{x\to 0} \frac{\left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots\right) - 1}{x^4} = \lim_{x\to 0} -\frac{1}{2!} + \frac{x^4}{4!} - \frac{x^8}{6!} + \cdots = -\frac{1}{2}$$

   (b) NOTE: Problem should have said "Find the first four <u>*nonzero*</u> terms of ..."

   Now, here $f(x) = \dfrac{x^2}{\sqrt{3}}(1+x/3)^{-1/2}$ and

$$\begin{aligned}(1+x/3)^{-1/2} &= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (x/3)^n = \binom{-\frac{1}{2}}{0} + \binom{-\frac{1}{2}}{1}\frac{x}{3} + \binom{-\frac{1}{2}}{2}\left(\frac{x}{3}\right)^2 + \binom{-\frac{1}{2}}{3}\left(\frac{x}{3}\right)^3 + \cdots \\ &= 1 - \frac{1}{2}\cdot\frac{x}{3} + \frac{(-1/2)(-1/2-1)}{2!}\cdot\frac{x^2}{9} - \frac{(-1/2)(-1/2-1)(-1/2-2)}{3!}\cdot\frac{x^3}{27} + \cdots \\ &= 1 - \frac{x}{6} + \frac{x^2}{24} - \frac{5x^3}{432} + \cdots\end{aligned}$$

   and so the first four nonzero terms of the Maclaurin series are,

$$\frac{x^2}{\sqrt{3}}(1+x/3)^{-1/2} \approx \boxed{\frac{x^2}{\sqrt{3}} - \frac{x^3}{6\sqrt{3}} + \frac{x^4}{24\sqrt{3}} - \frac{5x^5}{432\sqrt{3}}}$$

   (c) Since $\sin(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$, we have

$$x^3 \sin(3x) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+4}}{(2n+1)!}}$$

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2. (12 pts) Show all work and justify your answers. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n^2}$,

(a) For what values of x does this series converge conditionally?

(b) For what values of x does this series converge absolutely?

(c) For what values of x does this series diverge?

Solution:

(a) We do the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(x+2)^n} \right| = |x+2| \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} \overset{L'H}{=} |x+2|$$

and so $|x+2| < 1$ which implies $-3 < x < -1$.

Check endpoints:

($*$) If $x = -3$, we have $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2} \overbrace{=}^{a^n b^n = (ab)^n} \sum_{n=1}^{\infty} \frac{(1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a (absolutely) convergent p-series.

($*$) And if $x = -1$, we have $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which is absolutely convergent since $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

So the power series is NOT conditionally convergent for any values of x.

(b) The power series is absolutely convergent for $\boxed{-3 \le x \le -1}$ (see work from part (a).)

(c) The power series is divergent for $(-\infty, -3) \cup (-1, +\infty)$.

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3. (15 pts) Do the following series converge or diverge? Justify your answer, name any test that you use.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{\ln(3n)}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{4n}}{n^2-5}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(5n)}{7n}$

Solution:

(a) Note that $\displaystyle\lim_{n\to+\infty} \frac{\ln(n)}{\ln(3n)} \overset{L'H}{=} \lim_{n\to+\infty} \frac{1/n}{3/3n} = 1 \neq 0$, so the series diverges by the Divergence Test.

(b) Note that for large n we have $\displaystyle\frac{\sqrt{4n}}{n^2-5} \approx \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ and

$$\lim_{n\to\infty} \frac{\sqrt{4n}}{n^2-5} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{2n^2}{n^2-5} \overset{L'H}{=} 2.$$

Now, since $0 < 2 < \infty$ and $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series we have that $\displaystyle\sum_{n=3}^{\infty} \frac{\sqrt{4n}}{n^2-5}$ converges and so $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{4n}}{n^2-5}$ converges by Limit Comparison Test.

(c) Note that for $n \geq 1$ we have $\ln(5n) > \ln(e) = 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(5n)}{7n} > \sum_{n=1}^{\infty} \frac{1}{7n}$, now note $\displaystyle\sum_{n=1}^{\infty} \frac{1}{7n}$ is divergent since it is a non-zero constant multiple of the harmonic series, which is a divergent p-series. And so $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(5n)}{7n}$ diverges by Direct Comparison Test.

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4. (18 pts) Do the following series converge or diverge? If the series converges, does it converge **conditionally** or **absolutely**? Justify your answer, name any test that you use.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{4n-5}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{4^n}{n!}$ (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n!}{7n}$

Solution:

(a) Note that this is only an alternating series for $n > 1$. Now note $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{4n-5}\right| = \sum_{n=1}^{\infty} \frac{1}{4n-5}$, which is divergent by Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. Now, since $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{4n-5}$ is an alternating series with $b_n = \dfrac{1}{4n-5}$, which satisfies $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$, it is convergent and therefore $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4n-5}$ is also convergent, and so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4n-5}$ is conditionally convergent.

(b) Note that,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{4^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n} = \lim_{n\to\infty} \frac{4}{(n+1)} = 0 < 1$$

and so the series is absolutely convergent by the Ratio Test.

(c) Note that $\displaystyle\lim_{n\to\infty} (-1)^n \frac{n!}{7n} = \lim_{n\to\infty} (-1)^n \frac{(n-1)!}{7} \neq 0$, and so the series diverges by Divergence Test.

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5. (17 pts) Answer the following questions with as much detail as possible.

(a) (6 pts) What function does the series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^{3n+2}}{n!}$ represent?

(b) (6 pts) What number does the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converge to?

(c) (5 pts) If we use the first four terms of the series to approximate the number in part (b), what is the error bound? Is this approximation an *underestimate* or an *overestimate*? Justify your answer.

Solution:

(a) Note that $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^{3n+2}}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^{3n} x^2}{n!} = x^2 \sum_{n=0}^{\infty} \frac{(3x^3)^n}{n!} = \boxed{x^2 e^{3x^3}}$ since $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for all x.

(b) Since $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for all x, it follows that $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1}$.

(c) If we approximate $e^{-1} \approx 1 - 1 + \dfrac{1}{2!} - \dfrac{1}{3!}$, then by the Alternating Series Error Estimation Theorem we have $\boxed{|\text{error}| \leq \dfrac{1}{4!} = \dfrac{1}{24}}$, and since the first unused term was positive, we have an *underestimate.*

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6. (20 pts–4pts ea.) Answer "Always True" or "False" . **You do NOT need to justify your answer.**

(a) If the sequence $\{a_n\}_{n=1}^{\infty}$ converges to zero, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ will converge.

(b) If you show that $0 \leq b_n \leq a_n$ for all n, and the series $\displaystyle\sum_{n=1}^{\infty} b_n$ converges then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(c) If $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ are both divergent series, then $\displaystyle\sum_{n=1}^{\infty} a_n + b_n$ will also be divergent.

(d) If $\displaystyle\sum_{n=1}^{\infty} b_n 7^n$ converges then $\displaystyle\sum_{n=1}^{\infty} b_n(-5)^n$ also converges.

(e) If $0 \leq a_n \leq b_n$ for all n, and the sequence $\{b_n\}_{n=1}^{\infty}$ converges then the sequence $\{a_n\}_{n=1}^{\infty}$ also converges.

Solution:

(a) F (b) F (c) F (d) A.T. (e) F