APPM 2350 EXAM 1 SOLUTIONS SUMMER 2004

1. (25 points)
   (a) (5) $\text{proj}_{\mathbf{B}}\mathbf{A} = \frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{B}\cdot\mathbf{B}}\mathbf{B} = \frac{1}{9}\,(1\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$.
   (b) (5) $\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right) = \cos^{-1}\left(\frac{1}{3\sqrt{3}}\right)$ ($\approx 78.9\,\text{deg}$).
   (c) (5) Area$= \frac{1}{2}|\mathbf{A}\times\mathbf{B}| = \frac{1}{2}\,|-4\,\mathbf{i} + \mathbf{j} - 3\,\mathbf{k}| = \frac{\sqrt{26}}{2}$.
   (d) (5) $1(x-1) - 2(y-1) - 2(z-1) = 0 \iff x - 2y - 2z = -3$.
   (e) (5) We have $\vec{PB} = -3\,\mathbf{j} - 3\,\mathbf{k}$ and $\vec{n} = \mathbf{B}$. Hence distance$=\left|\frac{\vec{PB}\cdot\mathbf{B}}{|\mathbf{B}|}\right| = \frac{12}{3} = 4$.

2. (25 points) This is very similar to homework problem 11.4.15.
   (a) (5) Velocity$= \mathbf{v}(t) = t^2\,\mathbf{i} + t\,\mathbf{j}$, so speed$= |\mathbf{v}(t)| = t\sqrt{t^2+1}$ since $t \geq 0$.
   (b) (5) These points may be obtained when $t = 0$ and $t = 2$. Hence arc length $= \int_0^2 \tau\sqrt{\tau^2+1}\,d\tau = \frac{1}{2}\int_1^5 \sqrt{u}\,du = \frac{1}{3}\left(5^{3/2} - 1\right)$, using the substitution $u = t^2 + 1$.
   (c) (5) $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{t}{\sqrt{t^2+1}}\,\mathbf{i} + \frac{1}{\sqrt{t^2+1}}\,\mathbf{j}$.
   (d) (5) $\frac{d\mathbf{T}}{dt} = \frac{1}{(t^2+1)^{3/2}}\,\mathbf{i} - \frac{t}{(t^2+1)^{3/2}}\,\mathbf{j}$. Thus $\mathbf{N} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{t^2+1}}\,\mathbf{i} - \frac{t}{\sqrt{t^2+1}}\,\mathbf{j}$. (The previous part allows us to avoid another magnitude calculation.)
   (e) (5) $\mathbf{B} = \mathbf{T}\times\mathbf{N} = -\,\mathbf{k}$. $\mathbf{B}$ points directly into the page by convention (the right hand rule).

3. (23 points) Some questions about functions of two variables:
   (a) (4) Domain: All points in the xy plane. Range: $0 < z < \infty$.
   (b) (4) Level curves: Straight lines of the form $x + y = \ln(z)$, slope -1, and $z > 0$.
   (c) (5) $\lim\limits_{(x,y)\to(1,1)} \frac{x^3y^3-1}{xy-1} = \lim\limits_{(x,y)\to(1,1)} \frac{(xy-1)(x^2y^2+xy+1)}{xy-1} = \lim\limits_{(x,y)\to(1,1)} \left(x^2y^2 + xy + 1\right) = 3$.
   (d) (5) Considering the path $y = mx$, we have $L = \lim\limits_{\substack{(x,y)\to(0,0)\\ xy\neq 0, y=mx}} \frac{x^2+y^2}{xy} = \lim\limits_{\substack{(x,y)\to(0,0)\\ xy\neq 0, y=mx}} \frac{1+m^2}{m} = \frac{1+m^2}{m}$. For different values of m we can get different limits (ie.- $m = 1$ gives $L = 2$, $m = 2$ gives $L = \frac{5}{2}$), so the limit DNE by the two path test.
   (e) (5) $f_x = -\sin(x)\sin(y)$ and $f_y = \cos(x)\cos(y)$, so $L(x,y) = \frac{1}{2} - \frac{1}{2}\left(x - \frac{\pi}{4}\right) + \frac{1}{2}\left(y - \frac{\pi}{4}\right)$.

4. (15 points)
   (a) (5) Taking a partial derivative with respect to x gives $2xy + 3z^2\frac{\partial z}{\partial x}y - 2z - 2x\frac{\partial z}{\partial x} = 0 \iff \left(3z^2y - 2x\right)\frac{\partial z}{\partial x} = 2z - 2xy$. At $(-3,-1,-3)$, $\frac{\partial z}{\partial x} = \frac{4}{7}$.
   (b) (5) $dA = \frac{\partial A}{\partial a}da + \frac{\partial A}{\partial b}db = \pi b\,da + \pi a\,db = 200\pi\,da + 100\pi\,db$. More sensitive to a.
   (c) (5) $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \frac{e^y}{\sqrt{t}} + (xe^y + \sin(z))\left(1 + \frac{1}{t}\right) + (y\cos(z) + \sin(z))\,\pi$. At $t = 1$, $(x,y,z) = (2,0,\pi)$ and so $\frac{dw}{dt}(t=1) = 5$.

5. (12 points)
   (a) (2) (i)
   (b) (2) (v)
   (c) (2) (iv)
   (d) (2) (ii)
   (e) (2) (iii)
   (f) (2) (vi)