APPM 2350 FINAL EXAM EQUATIONS Summer 2008

Projections and distances

$$\mathrm{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Arc length, Frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$$

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

The Second Derivative Test

Suppose $f(x,y)$ and its first and second partial derivatives are continuous in a disk centered at (a,b) and $f_x(a,b) = f_y(a,b) = 0$. Let $D = f_{xx}f_{yy} - f_{xy}^2$.

1. If $D > 0$ and $f_{xx} < 0$ at (a,b), then f has a local maximum at (a,b).
2. If $D > 0$ and $f_{xx} > 0$ at (a,b), then f has a local minimum at (a,b).
3. If $D < 0$ at (a,b), then f has a saddle point at (a,b).
4. If $D = 0$ at (a,b), then the test is inconclusive.

Directional derivative, Discriminant Lagrange Multipliers, and Estimating Change

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} = D_{\mathbf{u}}f \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g \qquad df = (\nabla f|_{P_0}\cdot\mathbf{u})ds$$

Taylor's Formula (at the point (x_0,y_0))

$$\begin{aligned} f(x,y) \;=\; & f(x_0,y_0) + \Big((x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\Big) \\ & + \frac{1}{2!}\Big((x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\Big) \\ & + \frac{1}{3!}\Big((x-x_0)^3 f_{xxx}(x_0,y_0) + 3(x-x_0)^2(y-y_0)f_{xxy}(x_0,y_0) \\ & \qquad + 3(x-x_0)(y-y_0)^2 f_{xyy}(x_0,y_0) + (y-y_0)^3 f_{yyy}(x_0,y_0)\Big) + \cdots \end{aligned}$$

Linear Approximation Error, $E(x,y)$

$$|E(x,y)| \le \frac{1}{2}M(|x-x_0| + |y-y_0|)^2, \qquad \text{where } \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} \le M$$

Polar coordinates $x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2 \qquad dA = dx\,dy = r\,dr\,d\theta$

Cylindrical and spherical coordinates

Cylindrical to Rectangular	Spherical to Cylindrical	Spherical to Rectangular
$x = r\cos\theta$	$r = \rho\sin\phi$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$z = \rho\cos\phi$	$y = \rho\sin\phi\sin\theta$
$z = z$	$\theta = \theta$	$z = \rho\cos\phi$

$$dV = dx\,dy\,dz = rdz\,dr\,d\theta = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Substitutions in multiple integrals

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\left(x(u,v), y(u,v)\right)|J(u,v)|\,du\,dv \quad \text{where} \quad J(u,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Mass, first moments and center of mass

$$\begin{aligned} \text{Mass} \quad M &= \iiint_R \delta\,dV \\ \text{Moments} \quad M_{yz} &= \iiint_R x\,\delta\,dV \qquad M_{xz} = \iiint_R y\,\delta\,dV \qquad M_{xy} = \iiint_R z\,\delta\,dV \\ \text{Center of mass} \quad \bar{x} &= M_{yz}/M \qquad \bar{y} = M_{xz}/M \qquad \bar{z} = M_{xy}/M \end{aligned}$$

Flow and flux in a plane

$$\begin{aligned} \text{Flux} &= \int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C Mdy - Ndx = \int\int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA \\ \text{Flow} &= \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C Mdx + Ndy = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \end{aligned}$$

Green's Theorem in a plane (The curve C is traversed so that the region is on the left.)

$$\begin{aligned} \text{General Form} \quad & \oint_C \alpha\,dx + \beta\,dy = \iint_R \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right)\,dA \\ \text{Outward Flux} \quad &= \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,dA = \iint_R (\nabla\cdot\mathbf{F})\,dA \\ \text{Circulation} \quad &= \oint_C \mathbf{F}\cdot\mathbf{dr} = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dA = \iint_R (\nabla\times\mathbf{F})\cdot\hat{\mathbf{k}}\,dA \end{aligned}$$

Fundamental Theorem of Line Integrals
If $\mathbf{F}$ is conservative, then $\oint_C \mathbf{F}\cdot\mathbf{dr} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where $\mathbf{r}(t)$ parameterizes C for $a \le t \le b$, and $\mathbf{F} = \nabla f$.

Surface area of level surface $f(x,y,z) = c \qquad S = \iint_S d\sigma \quad$ where $\quad d\sigma = \dfrac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}dA$

Stoke's Theorem $\quad$ Circulation $= \oint_C \mathbf{F}\cdot\mathbf{dr} = \iint_S \text{curl}\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma$

Divergence Theorem of Gauss $\quad$ Flux $= \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \text{div}\mathbf{F}\,dV = \iiint_D \nabla\cdot\mathbf{F}\,dV$