INSTRUCTIONS: Electronic devices, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

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1. (25 points) Consider the function $f(x, y) = \dfrac{x^2}{2} + \dfrac{y^2}{2} - \dfrac{x^3}{3} - \dfrac{y^3}{3}$.

   (a) Determine the location of all critical points.

   (b) Classify all critical points and determine the value of $f(x, y)$ at each of these locations.

   (c) What are the local extreme values of f?

   (d) Assume you drew a straight line between the locations of the local extreme values of f, and looked at the midpoint along the line. At that location, what would be the maximum rate of increase in the value of f.

2. (25 points) Simon the cat is very lucky because his owners (called guardians in Boulder) built a narrow "cat walk" suspended a few feet below the ceiling. The shape of the cat walk is described by the curve $x^2 + y^2 = 2$. One day, in the plane of the cat walk, the room temperature is described by the function $T(x, y) = (x - 1)^2 + (y - 1)^2 + 70$. (Hint: you will only receive full credit for this problem if you use an appropriate technique involving gradients from Calc III!)

   (a) Simon loves to snooze at the warmest spot on the cat walk. On the day in question, what are the x and y coordinates where Simon is most likely to snooze?

   (b) Where are you least likely to find Simon sleeping?

   (c) What is the temperature at the locations you found in parts (a) and (b)?

3. (25 points) Suppose, on the day described in problem 2, Simon happens to be sleeping at the location $\left(\sqrt{3/2}, \sqrt{1/2}\right)$ on his cat walk. He wakes up, stretches, and starts to walk away from the location where he was sleeping at a speed of 2 units of distance per unit time, such that his x coordinate is decreasing. The following questions refer to the instant that he begins to walk away from where he was sleeping.

   (a) What is the unit vector in the direction of Simon's motion?

   (b) At what rate does he see the temperature changing with respect to time?

   (c) At what rate does he see the temperature changing with respect to distance traveled?

   (d) If Simon walks for 0.2 units of time, by approximately how much has the temperature changed?

   (e) Instead, if Simon walks for 0.2 units of distance, by approximately how much has the temperature changed?

4. (25 points) Consider the function $f(x, y) = x^3\, y^4$.

   (a) Calculate the *second order* Taylor approximation to $f(x, y)$ near the point $(1, 1)$.

   (b) Use your result from part (a) to estimate the value of $f(1.1, 1.1)$.

   (c) Calculate an "upper bound on the error" associated with this *second order* approximation assuming that you only use values of x and y such that $|x - 1| \leq 0.1$ and $|y - 1| \leq 0.1$.

Projections and distances

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$\begin{aligned} f(x,y) \quad = \quad & f(x_0,y_0) + \Big[\,(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\Big] \\ & + \frac{1}{2!}\Big[\,(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\Big] \\ & + \frac{1}{3!}\Big[\,(x-x_0)^3 f_{xxx}(x_0,y_0) + 3(x-x_0)^2(y-y_0)f_{xxy}(x_0,y_0) \\ & \qquad + 3(x-x_0)(y-y_0)^2 f_{xyy}(x_0,y_0) + (y-y_0)^3 f_{yyy}(x_0,y_0)\Big] + \cdots \end{aligned}$$

Linear approximation error

$$|E(x,y)| \le \frac{1}{2}M(|x-x_0| + |y-y_0|)^2, \qquad \text{where } \max\{\,|f_{xx}|,\,|f_{xy}|,\,|f_{yy}|\,\} \le M$$