Exam II Review

October 19, 2009

Domain, Range, etc. (12.1-12.2)

Define:

- Domain
- Range
- Interior Point
- Boundary Point
- Open
- Closed
- Bounded/Unbounded
- Limits/Continuity

Partial Derivatives and the Chain Rule!!! (12.3 & 12.5)

- How to take a Partial Derivative
- Find Total Derivatives
- Chain Rule

Questions

1. Let $F(u,v)$ be any function of two variables $u = x^2 + y^2$ and $v = \sin(x^2 + y^2)$. Show that $w = F(u,v)$ satisfies $y\frac{\partial w}{\partial x} = x\frac{\partial w}{\partial y}$.
2. Calculate the total differential of $f(x,y) = x^3 y^{-1/2}$.

Multivariable Derivatives

Define:

- The Gradient: At ever point (x_0, y_0) in the domain of $f(x,y)$...
  - The unit vector pointing in the direction of greatest increase is $\frac{\nabla f}{|\nabla f|}$.

- The derivative of the function in the direction of greatest increase is $|\nabla f|$.
- Any direction orthogonal to ∇f is a direction in which f does not change.

- Directional Derivative
- Tangent Planes
  - A tangent plane of a level surface at a point (P_0) can be written in terms of the gradient of the level surface, and a point it passes through. What is the relevant equation?
  - How are the tangent planes of a function related to the linearization of a function?
  -

Identifying & Classifiying Critial Points

The critical points of a function are found by solving $\nabla f|_{(x_o,y_o,z_o)} = \overrightarrow{0}$ for (x_o, y_o, z_o). The critical points can be classified in terms of second derivatives of the function at that point, (almost) just like in single variable calculus.

Local critical points	Multivariable	Single Variable
Maxima	$f_{xx} < 0$ and $f_{xx}f_{yy} > f_{xy}^2$	$f''x > 0$
Minima	$f_{xx} > 0$ and $f_{xx}f_{yy} -> f_{xy}^2$	$f''(x) < 0$
Saddle	$f_{xx}f_{yy} < f_{xy}^2$	not possible in 1D!
Inconclusive	$f_{xx}f_{yy} - f_{xy}^2 = 0$	$f''(x) = 0$

There are 3 places where a function may take on it's extreme values, so check all of them:

1. At local maxima and minima.
2. At points where the partial derivative does not exist.
3. On the boundary of the domain.

Questions

1. The temperature of a metal plate is given by $T(x, y) = x^2 + y^2 - x$
   (a) Identify and classify all critical points if the domain of T is the whole plane.
   (b) If the metal plate is bounded by the lines $y = 0$, $x = 0$ and $x+y = 1$, determine all possible extremizing points of $T(x, y)$ on the plate. It is not necessary to evaluate $T(x, y)$ at these points.
   (c) If the metal plate is cut into the shape described by $x^2 + y^2 = 1$, find all possible extremizing values of $T(x, y)$ on the plate. It is not necessary to evaluate $T(x, y)$ at these points.
2. How many critical points does the function $f(x, y, z) = \cos(x) + ye^x + xz$ have?
3. Consider the multivariable function $f(x, y) = x^4 + y^4 - 4xy + 1$. Find the (x, y) coordinates of all the local maxima, local minima, and saddle points for $f(x, y)$. For each point you

   find, be sure to clearly state which of these three types the point is.

Questions

1. What vector points in the direction of steepest ascent of the function$f(x, y) = c$?
2. Let $f(x, y, z) = z^3 + x^2z - 2xy$. Compute ∇f. If a function is implicitly defined by $f(x, y, z) = 2$, then evaluate f_x,f_y,f_{xy} at$(1, 0, 1)$. Find the equation of the tangent plane and the normal lime to the surface $f(x, y, z) = 2$ at $(1, 0, 1)$.

Langrange Muliplier Method

You wish to extremize (minimize or maximize) a funtion $f(x, y)$, subject to the constraint that your solution lie on the level set $g(x, y) = c$. In order to do this, use the Lagrange multiplier method.

1. Set up an alternate equation $\nabla f(x, y) = \lambda \nabla g(x, y)$.
2. Solve for λ.
3. Substitute λ into your constraint equation.
4. Solve for x and y.

Questions

1. Find the points on the surface $z^2 = xy + 4$ closest to the origin.
2. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.
3. The base of an open-top rectangular box costs $3 per square meter to construct, the the sides cost only $1 per square meter. Find the dimensions of the box of greatest volume that can be constructed for exactly $36.

Identifying & Classifiying Critial Points

What are the critical points of the function $f(x, y, z) = c$?

	Multivariable	Single Variable
Maxima	$f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$	$f''x > 0$
Minima	$f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$	$f''(x) < 0$
Saddle	$f_{xx}f_{yy} - f_{xy}^2 < 0$	
Inconclusive	$f_{xx}f_{yy} - f_{xy}^2 = 0$	$f''(x) = 0$

Approximations and Associated Errors (12.4 & 12.10)

- Taylor Series:

$$\begin{aligned} f(x, y) &= f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) \\ &+\frac{1}{2!}[(x - x_0)^2 f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0)f_{xy}(x_0, y_0) + (y - y_0)^2 f_{yy}(x_0, y_0)]... \end{aligned}$$

- Linerization, a truncated Taylor Series

$$\begin{aligned} f(x, y) &= f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) \\ & |E(x, y)| \le \frac{1}{2}M(|x - x_0| + |y - y_0|)^2 \end{aligned}$$

Question: How is M defined?

- Predict Change with Differentials

What it is called	How it is calculated	Keywords
Total error	Total differential df	absolute
Relative error	$\frac{df}{f(x_0,y_o)}$	relative
Percent Error	$\frac{df}{f(x_0,y_0)} \times 100$	percent
n^{th} order approximation	Taylor series!	Linearization, quadratic or cubic approximation (1^{st}, 2^{nd} or 3^{rd} term)

Questions

1. What is the definition(equation) of a total differential?
2. What is the linearization of $f(x, y) = \sin(x - y)$ near the origin? If we know that $-0.5 \leq x \leq 0.5$ and $-0.5 \leq y \leq 0.5$, what is the bound on the error of our approximation?
3. Suppose you are to evaluate the function $f(x, y, z) = x^2y^3 = z$, however you put in the wrong values for each variable. In particular, the value you use for each variable is 10% high. Estimate the relative error on the calculated value of f.
4. You are trying to estimate the volume, V of a circular cylinder by measuring the radius r and the height h. The relative error in the volume must be less than 0.01. Determine an upper bound on the relative errors of both the radius and the height if the relative error in the height is equal to twice the relative error in the radius.
5. The volume $V = \pi r^2 h$ of a right circular cylinder is to be calculated from measured values of $r$and h. Suppose that r is measured with an error of no more than 2%and$h$with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of V.