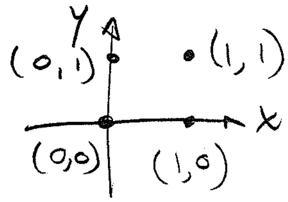
1

a) $f_x = x - x^2 = x(1-x)$

$f_y = y - y^2 = y(1-y)$ so C.P.'s are



b) $D = f_{xx}f_{yy} - (f_{xy})^2 = (1-2x)(1-2y) - (0)^2$

$D\big|_{(0,0)} = 1$ and $f_{xx}\big|_{(0,0)} = 1 \Rightarrow$ min of $f(0,0) = 0$

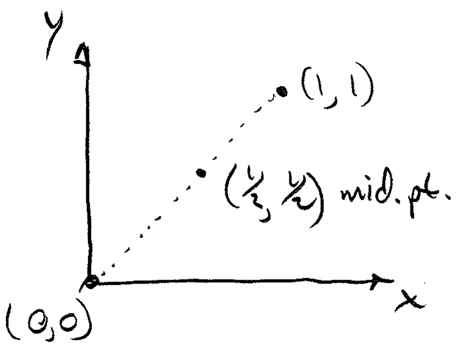
$D\big|_{(0,1)} = -1$ so saddle pt $f(0,1) = 1/6$

$D\big|_{(1,0)} = -1$ so Saddle pt $f(1,0) = 1/6$

$D\big|_{(1,1)} = 1$ and $f_{xx}\big| = -1$ so max $f(1,1) = 1/3$

c) $f(0,0) = 0$ min $f(1,1) = 1/3$ max

d)



$\nabla f = (x - x^2)\hat{i} + (y - y^2)\hat{j}$

so $\nabla f\big|_{(\frac{1}{2},\frac{1}{2})} = \frac{1}{4}\hat{i} + \left(\frac{1}{4}\right)\hat{j}$

but max rate of f increase $= |\nabla f|$

so $\left.\frac{df}{ds}\right|_{\substack{\max \\ \text{at } (\frac{1}{2},\frac{1}{2})}} = \left|\frac{1}{4}\hat{i} + \frac{1}{4}\hat{j}\right| = \sqrt{2}/4$

(2)

a) $T = (x-1)^2 + (y-1)^2 + 70$ obj.

$x^2 + y^2 = 2 = g(x,y)$ Const.

$\nabla T = \lambda \nabla g$ so $2(x-1)\hat{i} + 2(y-1)\hat{j} = \lambda[2x\hat{i} + 2y\hat{j}]$

∴ 3 eqns, 3 unknowns

(1) $x - 1 = \lambda x$

(2) $y - 1 = \lambda y$

(3) $x^2 + y^2 = 2$

from (1) + (2) $\lambda = \frac{x-1}{x} = \frac{y-1}{y}$ so $(x-1)y = (y-1)x$

or $y = x$

using (3) with $y=x$ gives $x^2 + x^2 = 2$ or $x^2 = 1$ + $y^2 = 1$

$x = 1$, $y = 1$ $\qquad$ $x = -1$, $y = -1$

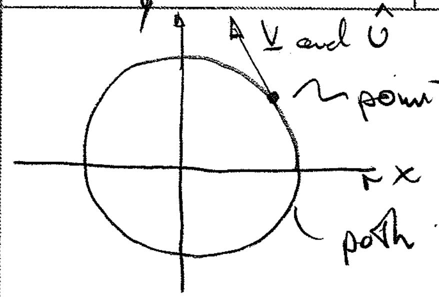
∴ $T(1,1) = 70$ $\qquad$ $T(-1,-1) = 78$

is min of T on $g = 2$ curve $\qquad$ is max of T on $g(x,y) = 2$ curve

∴ Simon is most likely to be at $(-1,-1)$ where $T = 78$

b) Simon is least likely to be at $(1,1)$ where $T = 70$

(3)



$T = (x-1)^2 + (y-1)^2 + 70$

$\nabla T = 2(x-1)\hat{i} + 2(y-1)\hat{j}$

$\nabla T\big|_p = 2\left(\sqrt{\frac{3}{2}}-1\right)\hat{i} + 2\left(\sqrt{\frac{1}{2}}-1\right)\hat{j}$

Note that the slope of the path is

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-2x}{2y} = \frac{-x}{y} \quad \text{so} \quad \frac{dy}{dx}\bigg|_p = \frac{-\sqrt{3/2}}{\sqrt{1/2}} = -\sqrt{3}$$

a) Thus $\hat{u} = \dfrac{-\hat{i} + \sqrt{3}\,\hat{j}}{2}$ (so Simon's x-coord decreases)

b) $$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = \left(\nabla T\big|_p \circ \hat{u}\right)\cdot|V|$$
$$= \left(2\left(\sqrt{\tfrac{3}{2}}-1\right)\hat{i} + 2\left(\sqrt{\tfrac{1}{2}}-1\right)\hat{j}\right)\circ\left(\frac{-\hat{i}+\sqrt{3}\,\hat{j}}{2}\right)\cdot 2$$
$$= 2\left(1-\sqrt{3}\right)$$

c) $$\frac{dT}{ds} = \nabla T\big|_p \circ \hat{u} = \left(1-\sqrt{3}\right)$$

d) $$\Delta T \approx \left(\frac{dT}{dt}\right)\cdot\Delta t = 2\left(1-\sqrt{3}\right)\cdot 0.2 = 0.4\left(1-\sqrt{3}\right)$$

e) $$\Delta T \approx \left(\frac{dT}{ds}\right)\Delta s = \left(1-\sqrt{3}\right)0.2$$

(4) $f = x^3y^4$

$f_x = 3x^2y^4$ $\qquad f_y = 4x^3y^3$

$f_{xx} = 6xy^4$ $\qquad f_{yy} = 12x^3y^2$ $\qquad f_{xy} = 12x^2y^3$

$f_{xxx} = 6y^4$ $\qquad f_{yyy} = 24x^3y$ $\qquad f_{xxy} = 24xy^3$ $\qquad f_{xyy} = 36x^2y^2$

so $f\big|_{(1,1)} = 1$

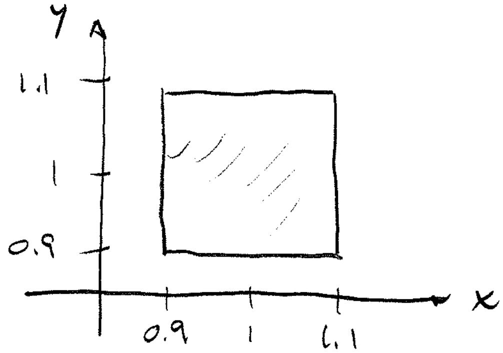
$f_x\big|_{(1,1)} = 3$ $\qquad f_y\big|_{(1,1)} = 4$

$f_{xx}\big|_{(1,1)} = 6$ $\qquad f_{yy}\big|_{(1,1)} = 12$ $\qquad f_{xy}\big|_{(1,1)} = 12$

$$f(x,y) \approx f\Big|_{(1,1)} + f_x\Big|_{(1,1)}(x-1) + f_y\Big|_{(1,1)}(y-1)$$
$$+ \frac{1}{2!}\left[f_{xx}\Big|_{(1,1)}(x-1)^2 + 2f_{xy}\Big|_{(1,1)}(x-1)(y-1) + f_{yy}\Big|_{(1,1)}(y-1)^2 \right]$$
$$= 1 + 3(x-1) + 4(y-1)$$
$$+ \frac{1}{2}\left[6(x-1)^2 + 2\cdot 12(x-1)(y-1) + 12(y-1)^2 \right]$$

b) so $f(1.1, 1.1) \approx 1 + 3(0.1) + 4(0.1)$
$$+ \frac{1}{2}\left[6(0.1)^2 + 24(0.1)^2 + 12(0.1)^2 \right]$$
$$= 1.91$$

c) Look in



for max mag of

$f_{xxx} = 6y^4$ $6(1.1)^4$

$f_{xyy} = 36x^2y^2$ $36(1.1)^4$

$f_{xxy} = 24xy^3$ $24(1.1)^4$

$f_{yyy} = 24x^3y$ $24(1.1)^4$

so pick $M = 36(1.1)^4$

Then
$$|\text{Error}| \leq \frac{M}{3!}\left(|x-x_0| + |y-y_0| \right)^3$$
$$= \frac{36(1.1)^4}{6}\left(0.1 + 0.1\right)^3 = 6(1.1)^4(0.2)^3$$