INSTRUCTIONS: Books, notes, crib sheets, and electronic devices are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show and explain your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (25 points) Determine the volume of material removed from the solid sphere $x^2 + y^2 + z^2 \leq 9$ by the cylinder $r = 3\cos\theta$.

2. (25 points) Consider the region in the xy-plane bounded by the closed curve $4x^2 + 4xy + 2y^2 + 4y = 5$. (It is actually an ellipse.) You need to find the enclosed area A. The substitution $u = 2x + y$ and $v = y + 2$ will simplify your calculation.

   (a) Find x and y in terms of u and v using the given substitution. Be sure to check this because the rest of the problem depends on this result!

   (b) Transform the original region R_{xy} into its corresponding region R_{uv} in the uv-plane. Make a clear sketch, of the new region of integration R_{uv} in the uv-plane. Be sure to label all axes, boundaries, intersection points, etc. on your sketch.

   (c) Rewrite the integral for A over the region R_{uv} in the uv-plane in terms of u and v.

   (d) Evaluate A in terms of u and v.

3. (25 points) The integral

$$V = \int_{\theta=0}^{2\pi} \int_{z=0}^{1} \int_{r=z}^{\sqrt{2}} r \ dr \ dz \ d\theta$$

   calculates the volume of an object.

   (a) Make a clear sketch of the cross-section of the object in a rz-plane (this is a constant θ plane in cylindrical coordinates) clearly labeling the bounding surfaces of the region of integration. (If you have trouble with this, you may "buy" a sketch of the shape of the region in the rz-plane for 5 points. **This sketch will only show the shape of the region**, so you will still need to supply the remaining details. The offer to buy this sketch ends at 6:15 PM!)

   (b) Express V in cylindrical coordinates using the order $dz\,dr\,d\theta$.

   (c) Express V in spherical coordinates using the order $d\rho\,d\phi\,d\theta$. (Hint: if needed, you should refer to particular angles as $\phi = \arctan(a/b)$. For example, $\phi = \arctan(\sqrt{2}/\sqrt{3})$.)

   (d) Express V in spherical coordinates using the order $d\phi\,d\rho\,d\theta$.

   (e) Evaluate one of the integrals above to determine the value of V.

4. (25 points) Consider the path (section 1) starting at the origin straight along the x-axis to the point $(3, 0)$, then (section 2) through the first quadrant along the curve $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ from the point $(3, 0)$ to the point $(0, 2)$, and the vector function given by $\mathbf{F} = -4y\,\mathbf{i} + 2x\,\mathbf{j}$.

   (a) Sketch the entire path and give a parametrization for each section of the path.

   (b) Calculate the **flow** along each section of the path C. Be sure to clearly state what the flow is for each section.

   (c) Calculate the **flux** along each section of the path C. Be sure to clearly state what the flux is for each section.

Projections and distances $\mathrm{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A}$ $\quad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$ $\quad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Polar coordinates $x = r\cos\theta \quad y = r\sin\theta \quad r^2 = x^2+y^2 \quad dA = dx\,dy = r\,dr\,d\theta$

Cylindrical and spherical coordinates

Cylindrical to Rectangular	Spherical to Cylindrical	Spherical to Rectangular
$x = r\cos\theta$	$r = \rho\sin\phi$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$z = \rho\cos\phi$	$y = \rho\sin\phi\sin\theta$
$z = z$	$\theta = \theta$	$z = \rho\cos\phi$

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Substitutions in multiple integrals

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\left(x(u,v),y(u,v)\right)\,|J(u,v)|\,du\,dv \quad \text{where} \quad J(u,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Mass, moments, and center of mass Mass $M = \iint_R \delta\,dA$

Moments $M_x = \iint_R y\,\delta\,dA \quad M_y = \iint_R x\,\delta\,dA$ Center of mass $\bar{x} = M_y/M \quad \bar{y} = M_x/M$

Flow and flux $\text{Flow} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot\mathbf{V}\,dt = \int_C \mathbf{F}\cdot\mathbf{dr} = \int_C M\,dx + N\,dy$

$$\text{Flux} = \int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C M\,dy - N\,dx$$