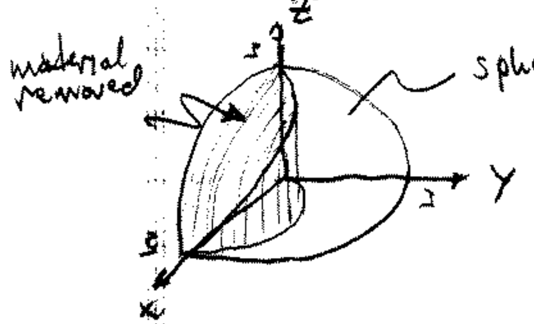
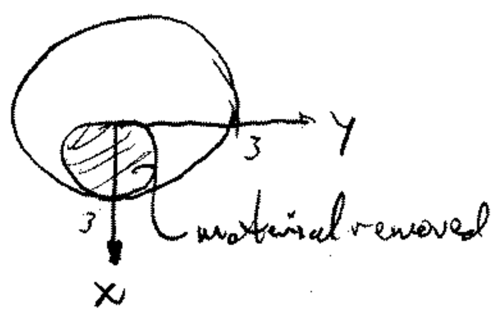
Fall 2009 Calc III, Exam 3

1. Looking at the 1^{st} octant   Looking down z axis



sphere is $x^2+y^2+z^2=9$
or $r^2+z^2=9$



The volume of material removed is

$$V = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{3\cos\theta}\int_{z=0}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{3\cos\theta} r\sqrt{9-r^2}\,dr\,d\theta$$

u-sub
$u = 9 - r^2$
$du = -2r\,dr$

$$= \left(-\tfrac{1}{2}\right)4\int_{\theta=0}^{\pi/2}\int_{u=9}^{9(1-\cos^2\theta)} \sqrt{u}\,du\,d\theta = \left(-\tfrac{1}{2}\right)4\left(\tfrac{2}{3}\right)\int_{\theta=0}^{\pi/2} 27\left((1-\cos^2\theta)^{\frac{3}{2}}-1\right)d\theta$$

$$= -(4\cdot 9)\int_{\theta=0}^{\pi/2}(\sin^3\theta - 1)\,d\theta$$

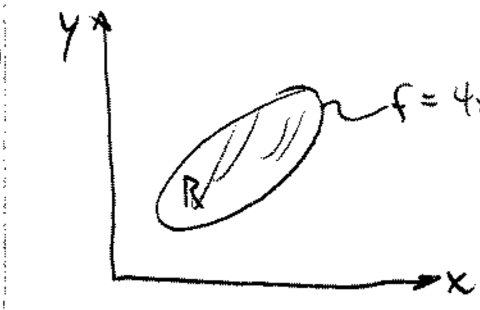
note $\sin^3\theta = (1-\cos^2\theta)\sin\theta$

$$= -36\int_{\theta=0}^{\pi/2}(\sin\theta - \cos^2\theta\sin\theta - 1)\,d\theta$$

$$= -36\left(-\cos\theta + \frac{\cos^3\theta}{3} - \theta\right)\Bigg|_0^{\pi/2}$$

$$= 18\pi - 24$$

2)



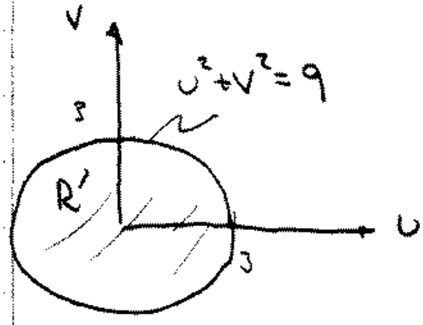
$f = 4x^2 + 4xy + 2y^2 + 4y = 5$

Curve is also

$$\underbrace{4x^2 + 4xy + y^2} + \underbrace{y^2 + 4y + 4} = 5 + 4$$

b)

$$f = (2x+y)^2 + (y+2)^2 = 9$$
$$= u^2 + v^2 = 9$$



So in u-v plane, the region of integration is a circle with radius 3

a) $u = 2x + y$ $\quad$ $x = \frac{u-y}{2} = \frac{u-v+2}{2} = \frac{1}{2}(u-v)+1$

$v = y + 2$ $\quad$ $y = v - 2$

c) $$J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

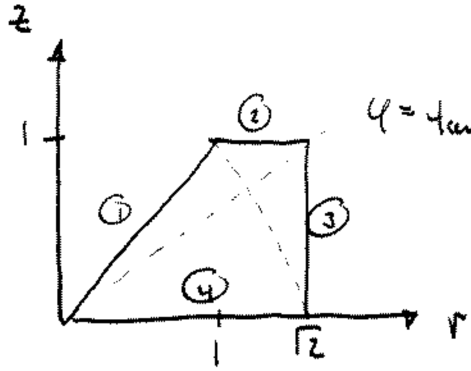
So $$A = \iint_R dx\,dy = \iint_{R'} |J(u,v)|\,du\,dv = \iint_{R'} \left|\tfrac{1}{2}\right| du\,dv$$

d) $$= \frac{1}{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{3} r\,dr\,d\theta = \frac{1}{2}\int_{\theta=0}^{2\pi}\left(\frac{9}{2}\right)d\theta$$

$$= \frac{1}{2}\left(\frac{9}{2}\right)2\pi$$

$$= \frac{9\pi}{2}$$

3

a) 

$\varphi = \tan^{-1}(\sqrt{2})$

Boundary sections are

	1	2	3	4
cyl	$z = r$	$z = 1$	$r = \sqrt{2}$	$z = 0$
sph	$\varphi = \pi/4$	$\rho\cos\varphi = 1$	$\rho\sin\varphi = \sqrt{2}$	$\varphi = \pi/2$

b) $$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{r} r\,dz\,dr\,d\theta + \int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{2}}\int_{z=0}^{1} r\,dz\,dr\,d\theta$$

c) $$V = \int_{\theta=0}^{2\pi}\int_{\varphi=\pi/4}^{\tan^{-1}(\sqrt{2})}\int_{\rho=0}^{\sec\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta + \int_{\theta=0}^{2\pi}\int_{\varphi=\tan^{-1}\sqrt{2}}^{\pi/2}\int_{\rho=0}^{\sqrt{2}\csc\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

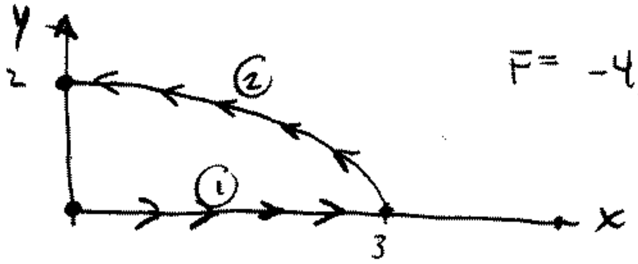
d) $$V = \int_{\theta=0}^{2\pi}\int_{\rho=0}^{\sqrt{2}}\int_{\varphi=\pi/4}^{\pi/2} \rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta + \int_{\theta=0}^{2\pi}\int_{\rho=\sqrt{2}}^{\sqrt{3}}\int_{\varphi=\cos^{-1}(1/\rho)}^{\sin^{-1}(\sqrt{2}/\rho)} \rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta$$

e) from original $$V = \int_{\theta=0}^{2\pi}\int_{z=0}^{1}\int_{r=z}^{\sqrt{2}} r\,dr\,dz\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{z=0}^{1}\left(1 - \frac{z^2}{2}\right)dz\,d\theta = \int_{\theta=0}^{2\pi}\frac{5}{6}\,d\theta = \frac{5}{3}\pi$$

4

a)



$F = -4y\,\hat{i} + 2x\,\hat{j}$

Path ①

$\underline{r}(t) = (3t)\,\hat{i} \quad (0 \le t \le 1)$

so $\underline{v}(t) = 3\,\hat{i}$

② $\quad 0 \le t \le \pi/2$

$\underline{r}(t) = 3\cos t\,\hat{i} + 2\sin t\,\hat{j}$

so $\underline{v} = -3\sin t\,\hat{i} + 2\cos t\,\hat{j}$

b) $$\text{flow}_{①} = \int_{①} \underline{F}\cdot\hat{T}\,ds = \int \underline{F}\cdot\underline{v}\,dt = \int_{t=0}^{1} -4y3\,dt \qquad (y=0 \text{ on } ①)$$

$$= \int_{t=0}^{1} -12\cdot 0\,dt = 0$$

$$\text{flow}_{②} = \int_{②} \underline{F}\cdot\underline{v}\,dt = \int (12y\sin t + 4x\cos t)\,dt$$

$$= \int_{t=0}^{\pi/2} (12\cdot 2\sin^2 t + 12\cos^2 t)\,dt = 12\int_{t=0}^{\pi/2} (1+\sin^2 t)\,dt$$

$$= 6\pi + \int_{t=0}^{\pi/2} 12\left(\frac{1-\cos 2t}{2}\right)dt = 6\pi + 3\pi - 6\left.\frac{\sin 2t}{2}\right|_{t=0}^{\pi/2}$$

$$= 9\pi$$

c) $$\text{flux}_{①} = \int_{①} \underline{F}\cdot\hat{n}\,ds = \int_{①} \big(\underbrace{-4\cdot 0\,\hat{i} + 2\cdot 3t\,\hat{j}}_{\underline{F}\text{ on }①}\big)\cdot\big(\underbrace{0\,\hat{i} - 3\,\hat{j}}_{\perp \text{ to } \underline{v}dt = \hat{n}ds}\big)\,dt$$

$$= \int_{t=0}^{1} -t18\,dt = -9$$

$$\text{flux}_{②} = \int_{②} \underline{F}\cdot\hat{n}\,ds = \int \big(\underbrace{-4\cdot 2\sin t\,\hat{i} + 2\cdot 3\cos t\,\hat{j}}_{\underline{F}\text{ on }②}\big)\cdot\big(\underbrace{2\cos t\,\hat{i} + 3\sin t\,\hat{j}}_{\hat{n}ds \text{ is } \perp \text{ to } \underline{v}dt}\big)\,dt$$

$$= \int_{t=0}^{\pi/2} (-16\sin t\cos t + 18\sin t\cos t)\,dt$$

$$= 2\int_{t=0}^{\pi/2} \sin t\cos t\,dt = \sin^2 t\Big|_{t=0}^{\pi/2} = 1$$