

Date: 21st January, 2010

1 Dot Product (a.k.a. Scalar Product)

It is an algebraic operation on two vectors and the output is always a scalar. In the context of euclidean vector space, you will most likely learn in more advanced engineering courses that this vector operator finds its use as an inner product, which may be used to define a norm of a mathematical object. But that may be stretching it too far for the purposes of this course.

Definition: The dot product of vectors \vec{a} and \vec{b} is given by $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$, where θ is the angle between the two vectors \vec{a} and \vec{b} .

Now as always one should ask what does it really mean ?

Physical/Geometrical interpretation:

Let us assume that \vec{b} is a unit vector i.e. $|\vec{b}| = 1$. Then $\vec{a} \cdot \vec{b} = |\vec{a}| \cos \theta$ gives us the scalar projection of \vec{a} onto (or in the direction of ... if the projection is +ve) \vec{b} . Figure 1 below should be helpful in this context.

The whole universe of engineering and science is replete with concepts based on projections and a thorough understanding

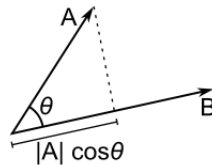


Figure 1: $|\vec{A}| \cos \theta$ is the scalar projection of \vec{A} onto \vec{B} .

of dot product cannot be more crucial. To drive home the intuition behind dot products, it may be useful to look at the following examples.

1. **Work done by a force:** Consider a particle on which a constant force F acts. Let the particle be given a displacement d . Then the work W done by F in the displacement is defined as the product of d and the component of F in the direction of d , that is,

$$W = |F| |d| \cos \alpha \quad (1)$$

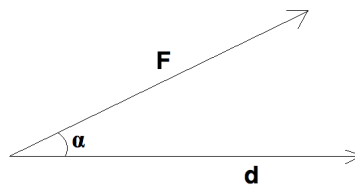


Figure 2: Work done by a force.

2. **Rotation of basis vectors:** *Don't get a panic attack if you do not follow the intricacies of this example completely!*

By now you should believe that $\{\hat{i}, \hat{j}, \hat{k}\}$ form a standard orthonormal basis (vector) in R^3 , meaning any vector in 3 dimensional space can be expressed in terms of $\hat{i}, \hat{j}, \hat{k}$. For simplicity, we will work only in 2 dimensional space to

illustrate the power of dot products in the context of rotation of basis vectors.

Let $B_1 = \{\hat{i}, \hat{j}\}$ and $B_2 = \{\hat{u}, \hat{v}\}$ be 2 different orthonormal (perpendicular and unit magnitude) bases of the same space, R^2 as illustrated in the Figure 3 below:

Let \exists a vector \vec{x} in this R^2 space. You should be able to convince yourself that \vec{x} can be expressed in terms of either

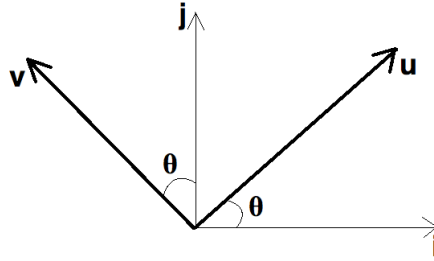


Figure 3: Rotation of orthogonal basis vectors.

B_1 or B_2 , in fact \vec{x} may be expressed in terms of any choice of the orthonormal basis that we make.

Now, let $\vec{v} = a\hat{i} + b\hat{j}$ represent \vec{x} in terms of B_1 . Now if we rotate B_1 and obtain a new orthonormal basis B_2 , an inquisitive mind may ask how can we express the vector \vec{x} in terms of the newly obtained orthonormal basis B_2 ? In other words, does there exist a vector $\vec{\mu} = \alpha\hat{u} + \beta\hat{v}$ that is a correct representation of the vector \vec{x} in terms of the new basis B_2 ? The simple answer is *Yes!* and is evident from the following:

$$\vec{\mu} = R\vec{v} = \begin{pmatrix} u_i & u_j \\ v_i & v_j \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \vec{u}_1 \cdot \vec{v} \\ \vec{v}_1 \cdot \vec{v} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\hat{u} + \beta\hat{v} \quad (2)$$

where $\vec{u}_1 = u_i\hat{i} + u_j\hat{j}$ and $\vec{v}_1 = v_i\hat{i} + v_j\hat{j}$ are the rotated basis vectors \hat{u}, \hat{v} in terms of the original basis B_1 .

Now, you may or may not know to do the matrix multiplication in equation (2) but it tells you that the same is equivalent to computing the respective dot products as is shown.

To further validate the power of the theory presented above, let us work out an actual example.

Let \exists a vector \vec{x} which is expressed in terms of B_1 as $\vec{v} = 2\hat{i} + 3\hat{j}$. The goal now is to express the same vector \vec{x} in terms of the rotated basis vectors (angle of rotation, $\theta = 45^\circ$) \hat{u}, \hat{v} . Clearly, (verify from figure above) $\hat{u} = (1/\sqrt{2})(\hat{i} + \hat{j})$ and $\hat{v} = (1/\sqrt{2})(-\hat{i} + \hat{j})$.

Therefore,

$$R = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

and hence

$$\vec{\mu} = R\vec{v} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \vec{u}_1 \cdot \vec{v} \\ \vec{v}_1 \cdot \vec{v} \end{pmatrix} = \begin{pmatrix} 5/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = (1/\sqrt{2})(5\hat{u} + \hat{v})$$

is the vector \vec{x} in terms of the new rotated basis B_2 !

3. **Orthogonal straight lines in a plane:** You will learn about **Lines and Planes in space**, section (10.5) in lecture tomorrow. Think about the following optional exercise over the weekend to further your grasp over dot products.

Optional fun exercise:

- (a) Find a representation (i.e. equation) of the straight line L_1 through the point $P : (1, 3)$ in the x-y plane and perpendicular to the line L_2 represented by $x - 2y + 2 = 0$. (Hint: Draw the figure and then proceed ... ☺)

2 Optional fun exercises from Handout 1:

1. For any vectors \vec{u} and \vec{v} , show that $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$
This should have been very easy to show! This result is known as the much celebrated *Cauchy - Schwartz inequality*.

2. For any vectors \vec{u} and \vec{v} , show that $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Proof:

$$\begin{aligned}
 (\|\vec{u} + \vec{v}\|)^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\
 &= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\
 &= \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \\
 &\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 \text{ (by Cauchy - Schwartz)} \\
 &= (\|\vec{u}\| + \|\vec{v}\|)^2
 \end{aligned}$$

... and the result follows immediately ! This result is more popularly known as the *triangular inequality*.

3 Homework discussion:

Problems: Section (10.3): 16, 29, 35

4 Miscellaneous topics of importance from section (10.3):

1. $Proj_{\vec{a}}\vec{b} = \left(\frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \right) \vec{a}$
2. **A vector as a sum of 2 orthogonal vectors:** $\vec{b} = Proj_{\vec{a}}\vec{b} + (\vec{b} - Proj_{\vec{a}}\vec{b})$, where the 1st term on the R.H.S. is parallel to \vec{a} and the 2nd term on the R.H.S. is orthogonal to \vec{a} .

5 Reading Assignment:

1. page 808-809
2. section (10.4): Cross Product (a.k.a. Vector product)
(I highly recommend that you prepare self notes on this section. You may use the structure of Handout 2 notes on Dot product and your textbook as a **basis** for the notes you prepare for Section (10.4))

Concluding remarks: From next week the handouts will feature a new section on **Past Exam Problems** from the week's topics !