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1 Lines in Space

We want to find the equation of a line, L through a point, $P_0 = (x_0, y_0, z_0)$ parallel to a given vector, $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$. We know that if a vector is parallel to another vector then one can be expressed as a scalar multiple of the other ! We assume that $P = (x, y, z)$ is any point on the line, L. Therefore,

$$\vec{P_0P} = t\vec{v}; \quad \forall t \in (-\infty, \infty)$$

Now, we need to find an expression in terms of (x, y, z), the given point (x_0, y_0, z_0) and the given vector $\vec{v}$.
$\vec{P_0P}$ can be expressed as $(x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}$ and $t\vec{v}$ can be written as $t(a\hat{i} + b\hat{j} + c\hat{k})$. Comparing the components along the basis vectors on both sides of the above equation, we derive the following expression for the equation of a line passing through the point P_0 and parallel to the given vector $\vec{v}$;

$$(x - x_0) = ta, \quad (y - y_0) = tb, \quad (z - z_0) = tc$$

which can be re-arranged and expressed as follows,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, \quad \text{for any } t \in (-\infty, \infty) \tag{1}$$

1.1 Optional fun exercise from Handout 2

1. Find a representation (i.e. equation) of the straight line L_1 through the point $P : (1, 3)$ in the x-y plane and perpendicular to the line L_2 represented by $x - 2y + 2 = 0$. (*Hint: Draw the figure and then proceed ...* ☺)

Soln:- Here, $P_0 = (x_0, y_0) = (1, 3)$. The given line, $L_2 : x-2y+2 = 0$ can be expressed as $\frac{x-(-2)}{1} = \frac{y}{(1/2)}$ from which we know that this line, L_2 is parallel to the vector $\vec{v_2} = \hat{i} + \frac{1}{2}\hat{j}$.
Remember, we are looking for a line, L_1 perpendicular to the line L_2; hence we must first look for a (any) vector $\vec{v_1}$ which is orthogonal (perpendicular) to the vector $\vec{v_2}$, i.e. we must have $\vec{v_1}.\vec{v_2} = 0$.
Now, you know it is not difficult to find such a vector, right ? Verify that $\vec{v_1} = \frac{1}{2}\hat{i} - \hat{j}$ is a suitable candidate. (check that) $\vec{v_1}.\vec{v_2} = 0$ is indeed true !
Now, all remains to find is the equation of the line through the point $P_0 = (1, 3)$ and parallel to the vector $\vec{v_1} = \frac{1}{2}\hat{i} - \hat{j}$. Therefore, we unleash the power of equation (1) and obtain $\frac{x-1}{(1/2)} = \frac{y-3}{-1}$ or equivalently $y = -2x + 5$, which is the equation of the line L_1. `Yahooooo .....   !!!`

1.2 Distance formula (lines)

The distance from a point S to a line through P parallel to $\vec{v}$ is given by:

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

Reading assignment: *Review example 4 from your textbook, page 823*

2 Planes in Space

Two important things to know are:

1. **eqn. of a plane through** $P_0 = (x_0, y_0, z_0)$ **and normal (orthogonal, perpendicular ?) to** $\hat{n} = a\hat{i}+b\hat{j}+c\hat{k}$

$$\hat{n}.\vec{P_0P} = 0$$

2. **Normal to a given plane**

$$a\hat{i} + b\hat{j} + c\hat{k} \text{ is normal to the plane } ax + by + cz = d$$

Reading assignment: Review example 6,7,8 page 825 from the text.

Distance formula (Planes) Note: to find the distance from a point, S to a given plane $ax + by + cz = d$, always begin by drawing a picture and step through what you need to find.

- find a point, P on the plane. *hint:* `find the intercepts, i.e. for eg. plug in` $x = 0, y = 0$ `in the eqn of the plane and find the point` z_0`, then a point on the plain is` $(0, 0, z_0)$
- find the vector $\vec{PS}$
- find the normal to the plane using the 2nd formula given above in this section ($\hat{n} = a\hat{i} + b\hat{j} + c\hat{k}$)
- now compute the distance, d by finding the magnitude of the projection of the vector $\vec{PS}$ on $\hat{n}$
  `(Yes ! I will not give you the formula because I want you to appreciate the beauty of mathematical concepts and not encourage you to become an` ∞ `GB computer RAM ☺)`

3 Sample review problems:

1. *Hint:* `Angle bet'n 2 planes is the angle bet'n the normals to the planes !`
   Given the equation of 2 planes, $P_1 : A_1x + B_1y + C_1z = D_1$ and $P_2 : A_2x + B_2y + C_2z = D_2$ and define θ as the angle between the planes, prove that $\cos\theta = \frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$
   Instead, if I tell you that the planes are parallel; obtain the relationship between $A_1, A_2, B_1, B_2, C_1, C_2$?

2. Find the equation of the plane containing the point $(2, -2, -7)$ and the line $\frac{x-1}{-2} = \frac{y-2}{1} = \frac{z-7}{7}$.

4 Vector Valued Functions and Space Curves

We have seen that for a static point (x, y, z) in space, the position vector is given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. But, what if this static guy (point) starts moving ? What is its position vector ? Well, it should not be hard to believe that the position vector now becomes a function of time, t !

$$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

Reading assignment: page 856-858 from your textbook.

Having defined the position vector $\vec{r}(t)$ of a point moving in space, wouldn't it be fun to now ask – *hey ! what is the velocity with which this point is moving ?* or *what is its acceleration ?*

1. Velocity, (rate of change !) $\vec{v}(t) = \frac{d\vec{r}}{dt}$, is tangent to the curve $\vec{r}(t)$ i.e. in the direction of instantaneous motion.
2. Acceleration, $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt}$

Note: - Velocity = $|\vec{v}|\left(\frac{\vec{v}}{|\vec{v}|}\right)$ = (speed)(direction)

Lemma: If $\vec{u}(t)$ is differentiable and $|\vec{u}(t)|$ = constant, then

$$\vec{u}.\frac{d\vec{u}}{dt} = 0$$

Application: If you are an electrical engineer you might have heard about AC machines ! One such machine, the **Rogowski coil** is an electrical device for measuring alternating current (AC) or high speed current pulses. It consists of a helical coil of wire with the lead from one end returning through the centre of the coil to the other end, so that both terminals are at the same end of the coil. The whole assembly is then wrapped around the straight conductor whose current is to be measured. Since the voltage that is induced in the coil is proportional to the rate of change (derivative) of current in the straight conductor, the output of the Rogowski coil is usually connected to an electrical (or electronic) integrator circuit in order to provide an output signal that is proportional to current.
The voltage produced by a Rogowski coil is $\vec{V}(t) = \kappa\frac{d\vec{I}}{dt}$, where κ is an electrical constant.
If the 2-phase AC current flowing through the modified Rogowski coil is given by $\vec{I}(t) = \sin t\hat{i} + \cos t\hat{j}$ implying that the 2-phase current is 90^0 out of phase; the voltage is given by $\vec{V}(t) = \kappa(\cos t\hat{i} - \sin t\hat{j})$. It is easy to check that $|\vec{I}(t)|$ = constant and hence using the lemma or by computing the dot product it is easy to check $\vec{I}.\vec{V} = \vec{I}.\frac{d\vec{I}}{dt} = 0$, implying that the voltage produced by the Rogowski coil (in this case) is 90^0 out of phase w.r.t. the current flowing through the coil.

5 Optional fun exercise

5.1 Past Exam Problems

1. (*Exam 1, Fall 2008, Q2*)
   Consider a position vector $\vec{r}(t)$ with a velocity vector that satisfies$|v(t)| = 1$.
   (a) Find the arc length of $\vec{r}(t)$ from $t = t_0$ to $t = t_1$
   (b) Show that the acceleration $\vec{a}(t)$ is orthogonal to the velocity $\vec{v}(t)$
   (c) We can decompose the acceleration into a sum of two components: $\vec{a}(t) = \vec{a_T}(t)\vec{T} + \vec{a_N}(t)\vec{N}$ where $\vec{a_T}(t)\vec{T}$ is the component of acceleration in the tangent direction and $\vec{a_N}(t)\vec{N}$ is the component of the acceleration in the normal direction. Give simple expressions for $\vec{a_T}(t)\vec{T}$ and $\vec{a_N}(t)\vec{N}$

Sketch of solutions to sample problems:

1. Proof should be easy if you follow the hint.
   If P_1 is parallel to P_2, then $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$. Why ? ...... `think think ...  mink !`

2. First find any 2 convenient points on the given line , say $(1, 2, -7)$ with $t = 0$ and $(5, 0, -7)$ with $t = -2$. Now use the fact that the required plane $ax + by + cz = d$ contains the points $(1, 2, -7)$, $(5, 0, -7)$ and the given point $(2, -2, -7)$ to obtain the following 3 equations:

$$a + 2b + 7c = d$$
$$5a + 0b - 7c = d$$
$$2a - 2b - 7c = d$$

   and then solve for a, b, c to find $a = \frac{2d}{3}$, $b = -d$ and $c = \frac{d}{3}$ and hence the required plane is $2x - 3y + z = 3$