Date: 4^{th} February, 2010

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I highly recommend you to use the notes from this handout in conjugation with
the questions from quiz 1 to prepare for Exam 1.
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1 Arc Length and unit Tangent vector

We will only briefly touch upon the highlights from this section and discuss in detail the next section on **TNB** frame.

1.1 Length of a curve and arc length

1. **Length of a smooth curve** $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$, $a \le t \le b$ is:

$$L = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt \tag{1}$$

   or equivalently,

$$L = \int_a^b |\vec{v}(t)| dt \tag{2}$$

2. **Arc length:** The directed distance along the curve from $P(t_0)$ to any point $P(t)$ is :

$$s(t) = \int_{t_0}^t |\vec{v}(\tau)| d\tau \tag{3}$$

1.2 Unit Tangent vector, $\hat{T}$

1. **Speed with which the particle moves along the path is given by:**

$$\frac{ds}{dt} = |\vec{v}(t)|$$

2. The **unit tangent vector** of a differentiable curve $\vec{r}(t)$ is

$$\hat{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{\vec{v}}{|\vec{v}|} \tag{4}$$

   $\hat{T}$ is a unit vector in the direction of the tangent to the curve, $\vec{r}(t)$ i.e. $\hat{T}$ is in the direction of the velocity vector, $\vec{v}(t)$.

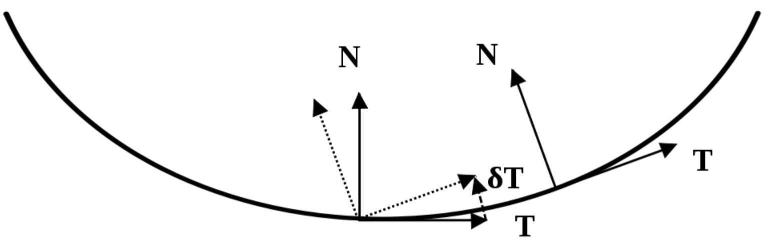

Figure 1: The **T** and **N** vectors at two points on a plane curve, a translated version of the second frame (dotted), and the change in **T**: $\delta\mathbf{T}$ Also note, δs is the distance between the points. In the limit $\frac{d\mathbf{T}}{ds}$ will be in the direction N and the curvature describes the speed of rotation of the frame.

2 Curvature, Torsion and the TNB frame

We will now gradually work our way through constructing a new set of orthonormal basis to describe the kinematic properties of a particle moving along a continuous, differentiable curve in three-dimensional Euclidean space, R^3. Let us begin with a few definitions.

2.1 Curvature, κ

As the name suggests, κ is defined as the *rate at which $\hat{T}$ turns (or bends) per unit of length along the curve*.

$$\kappa = \left|\frac{d\hat{T}}{ds}\right| \tag{5}$$

As shown in figure (1), it should not be hard to believe that curvature describes the speed of rotation of the frame. Later, we will define another property of a space curve viz. **torsion**. It may be useful to know that *curvature* and *torsion* of a space curve are analogous to the *curvature* of a plane curve. We will have to say more on this later but the point here is to encourage you to start appreciating the subtle geometrical manifestations of a space curve as compared to a plane curve.
Another useful formulation for *curvature* is

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

2.2 Unit Normal vector for Plane/Space curves

Note, that $\hat{T}$ being of unit (and hence constant) magnitude, we can again unleash the power of the ***lemma** in Handout 3* to conclude that $\frac{d\hat{T}}{ds} \cdot \hat{T} = 0$ i.e. to say that $\frac{d\hat{T}}{ds} \perp \hat{T}$. Now, it is possible to define a unit vector $\perp \hat{T}$ as follows:

$$\hat{\mathbf{N}} = \frac{d\hat{T}/ds}{|d\hat{T}/ds|} = \frac{\frac{\mathbf{d\hat{T}}}{\mathbf{ds}}}{\kappa} = \frac{d\hat{T}/dt}{|d\hat{T}/dt|} \tag{6}$$

By now you should appreciate that for a body moving in a plane, it may be convenient to represent the position, velocity, acceleration, etc in terms of tangential and normal basis vectors rather than the usual standard orthonormal basis $\{\hat{i}, \hat{j}\}$. Likewise, in polar coordinates, one may wish to use another set of ON basis viz. *Radial* (in the direction of the radius/position vector, $\vec{r}(t)$) and *Transverse* ($\perp$ Radial) bases (figure(2). Note these are not quite the same as $\hat{T}$ and $\hat{N}$ bases !

2.3 Sample review problems

1. Find the radial and transverse acceleration of a body moving in a plane curve. (`hint: use polar coordinates`)

   Soln:- At any time t, let the position vector of the moving body $P(r, \theta)$ be $\vec{R}(t)$. We define, $\hat{U}$ to be a unit vector $\perp \vec{R}(t)$. So, $\vec{R} = r\hat{R} = r(\cos\theta\hat{i} + \sin\theta\hat{j})$, where $\hat{R}$ is the unit vector in the direction of $\vec{R}$. (*verify this from the Figure 2* !)
   Note from the figure: $\hat{R} = \cos\theta\hat{i} + \sin\theta\hat{j}$; clearly $|\hat{R}| = 1$ (constant).
   $\frac{d\vec{R}}{dt} = (-\sin\theta\hat{i} + \cos\theta\hat{j})\frac{d\theta}{dt}$. Again using the **lemma** from handout 3, we conclude $\frac{d\hat{R}}{dt} \perp \hat{R}$.
   Also note $|d\hat{R}/dt| = |1|.(d\theta/dt)$. Hence we have shown $d\hat{R}/dt = (d\theta/dt)\hat{U}$.
   Now, recall that we defined $\vec{R} = r\hat{R}$. Therefore,

$$\vec{v}(t) = \frac{d\vec{R}(t)}{dt} = (dr/dt)\hat{R} + r(d\hat{R}/dt) \qquad \text{(product rule for differentiation)} \tag{7}$$

$$= (dr/dt)\hat{R} + r(d\theta/dt)\hat{U} \tag{8}$$

$$\begin{aligned}\vec{a}(t) &= \frac{d\vec{v}(t)}{dt} \\ &= (d^2r/dt^2)\hat{R} + (dr/dt)(d\hat{R}/dt) + (dr/dt)(d\theta/dt)\hat{U} + r(d^2\theta/dt^2)\hat{U} + r(d\theta/dt)(d\hat{U}/dt)\end{aligned}$$

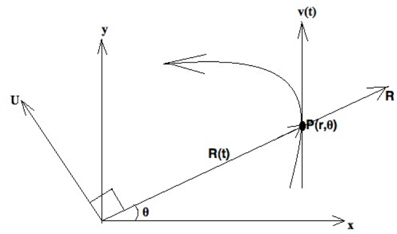

Figure 2: Radial, $\mathbf{R}$ and transverse, $\mathbf{U}$ direction pertaining to the curve traced by $P(r, \theta)$. Note that $\mathbf{U} \perp \mathbf{R}$

Therefore, re-arrange terms to obtain:
radial component of $\vec{a}$ to be = $\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2$
transverse component of $\vec{a}$ to be = $2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$

2.4 Tangential and Normal components of acceleration

Inspired by the **mathema$\mathcal{G}$ic** from the above sample problem we now state a very general result !

$$\vec{a} = a_T\hat{T} + a_N\hat{N} \tag{9}$$

where,

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\vec{v}| \tag{10}$$

$$a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\vec{v}|^2 \tag{11}$$

are the **tangential** and **normal** scalar components of acceleration. Therefore;

$$a_N = \sqrt{|\vec{a}|^2 - a_T^2} \tag{12}$$

Reading Assignment: *review Ques (2) from Quiz 1, example (4) (page-884) from textbook*

2.5 TNB frame

Recall, that our final and most important goal in this section was to construct an alternative orthonormal basis to represent kinematics of moving objects. We have already seen in much detail, the construction and utility of $\hat{T}$, $\hat{N}$ bases.
To wrap up on this new set of basis vectors, we now present the **binormal vector**, $\hat{B}$, which is a unit vector $\perp$ $\hat{T}$ and $\hat{N}$.

$$\hat{B} = \hat{T} \times \hat{N} \tag{13}$$

The Figure (3) below should serve as an appropriate self-explanatory theme on the discussion of the construction of the TNB frame !

2.5.1 Applications:

1. ... in models of microbial motion.
2. Relativity theory:- modelling precision of a gyroscope in a gravitational well.
3. Motion of spacecraft

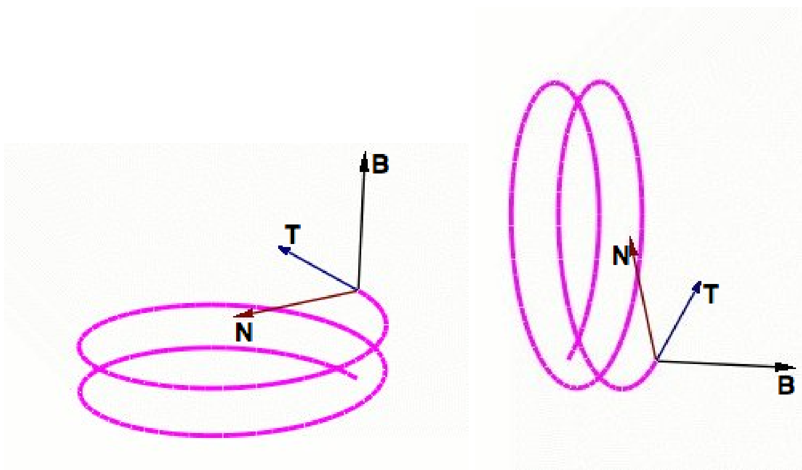

Figure 3: The Frenet-Serret (**TNB**) frame moving along a helix. **B** is the binormal vector $\perp$ vectors **T** and **N**. Unit vectors **T** and **N** comply with the description in Figure (1).

2.6 Torsion:

We had briefly stated that *curvature* together with *torsion* (measure of how sharply the space curve is **twisting**) play a vital role in the geometry of a space curve. We list 2 of the many properties of a space curve that depend on curvature and torsion to give a better intuition.

1. A plane curve with non-vanishing curvature has zero torsion at all points. Conversely, if the torsion of a regular curve is identically zero then this curve belongs to a fixed plane.
2. The curvature and the torsion of a **helix** are constant. Conversely, any space curve with constant non-zero curvature and constant torsion is a helix. The torsion is positive for a right-handed helix and is negative for a left-handed one.

Let us now formally define ***torsion***, τ .

$$\tau = -\frac{d\hat{B}}{ds} \cdot \hat{N} \tag{14}$$

One can verify the following result which is relevant in the above context:

$$\frac{d\hat{B}}{ds} \perp \text{ both } \hat{T} \text{ and } \hat{B}$$

Physical interpretation: The torsion, τ is the rate at which the **osculating plane** turns about $\hat{T}$ as the body/particle moves along the curve $\vec{r}(t)$. Torsion measures how the curve twists !

Note: ***Normal plane:*** *plane containing $\hat{N}$ and $\hat{B}$;* ***Osculating plane:*** *plane containing $\hat{N}$ and $\hat{T}$.*

Reading Assignment: *review the meaning/definition of* ***osculating circle*** *(page-884) from your textbook.*

2.7 Sample review problems:

1. Prove that if torsion, $\tau \equiv 0$, then the curve lies in a plane and this is the osculating plane at every point of the curve. In addition, if the curvature, κ is a non-zero constant; then show that the curve is a circle of radius $\frac{1}{\kappa}$. (hint: use the claim that $\hat{N}' = -\kappa\hat{T} + \tau\hat{B}$ is indeed true !)
(*This problem illustrates in detail the essence of the discussion on Circle of Curvature and radius of Curvature, page 884 in the textbook*)

Soln: $\tau \equiv 0 \implies \hat{B}' \equiv 0 \implies \hat{B}$ is constant. Let $G(s_0)$ be *any fixed* point on the curve; we want to show that the curve lies in the osculating plane

$$\{\vec{P} : (\vec{P} - \vec{G}(s_0)) \cdot \hat{B} = 0\}$$

why is the above true ? because then we would have shown that every point on the curve is $\perp \hat{B}$ and hence in the plane $\perp \hat{B}$!
Proving the above is equivalent to showing that the function $d(s) := (\vec{G}(s) - G(\vec{s_0})) \cdot \hat{B} = 0$ for every s. Clearly, $d(s_0) = 0$ and then differentiating $d'(s) = \vec{G}'(s) \cdot \hat{B} =$ (constant)$\hat{T}(s) \cdot \hat{B} = 0$ (b/c $\hat{T} \perp \hat{B}$ always !) Therefore, $d \equiv 0$.
To prove the second part, we need to show that the curve (which we proved above is restricted to a plane) is a circle of radius $\frac{1}{\kappa}$ i.e. we need to find a fixed point $\vec{C}$ s.t.

$$|\vec{C} - \vec{G}(s)| = \frac{1}{\kappa} \quad \forall s$$

Intuitively, $\vec{C}$ should be the centre of curvature; so it is natural to look at the function

$$\vec{C}(s) = \vec{G}(s) + \frac{1}{\kappa}\hat{N}(s) \tag{15}$$

Differentiating the above and using the hint along with the fact that κ is a constant and $\tau \equiv 0$; we obtain

$$\vec{C}'(s) = \vec{G}'(s) + \frac{1}{\kappa}\hat{N}'(s) = \hat{T}(s) - \hat{T}(s) = 0$$

which implies $\vec{C}$ is a constant vector and from equation(15) above we have

$$|\vec{C} - \vec{G}(s)| = \left|\frac{1}{\kappa}\hat{N}(s)\right| = \frac{1}{\kappa}$$

Hence proved !

3 Applications: *No panic zone !!!*

`Warning: This section is purely for motivational purposes and does not cover any exam material !`

1. Using equations (3), (4), (6) and (13), it is not difficult to obtain the following differential equations for the **TNB** frame, which are also known as the celebrated **TNB formulae or the *Frenet-Serret theorem***:

$$\frac{d\hat{T}}{ds} = \kappa\hat{N} \tag{16}$$
$$\frac{d\hat{N}}{ds} = -\kappa\hat{T} + \tau\hat{B} \tag{17}$$
$$\frac{d\hat{B}}{ds} = -\tau\hat{N} \tag{18}$$

which in matrix notation is quite convenient to remember as follows:

$$\begin{pmatrix} \hat{T}' \\ \hat{N}' \\ \hat{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \hat{T} \\ \hat{N} \\ \hat{B} \end{pmatrix}$$

Note, that the matrix on the R.H.S. is *skew-symmetric* !

2. In differential geometry, esp. the theory of space curves, the **Darboux vector**, $\vec{\omega}$ is the *areal velocity vector* (`:= rate at which area is swept out by a particle as it moves along a curve` Figure (4)) of the **TNB** frame of a space curve ! By the way, $\vec{\omega}$ is what you may know from your Physics class as the *angular momentum vector*. The Darboux vector, $\vec{\omega}$ in terms of the **TNB** apparatus is defined as, $\vec{\omega} = \tau\hat{T} + \kappa\hat{B}$, which is used to define the Darboux formulae:

$$\frac{d\hat{T}}{ds} = \omega \times \hat{T}; \quad \frac{d\hat{N}}{ds} = \omega \times \hat{N}; \quad \frac{d\hat{B}}{ds} = \omega \times \hat{B}$$

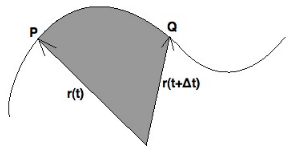

Figure 4: $\vec{\omega}$ is the rate at which the area of the shaded part changes

4 Exercise problems:

1. A ladybug is climbing on a Volkswagen Bug (:= VW). In its starting position, the surface of the VW is represented by the unit semi-circle $x^2 + y^2 = 1, y \geq 0$ in the xy-plane. The road is represented as the x-axis. At time $t = 0$ the ladybug starts at the front bumper, $(1, 0)$ and walks counter-clockwise around the VW at unit speed relative to the VW. At the same time the VW moves to the right at speed 10 units/time.
   (a) Find the parametric formula for the trajectory of the ladybug, and find it's position when it reaches the rear bumper. (At $t = 0$, the rear bumper is at $(-1, 0)$.)
   (b) Compute the speed of the bug and find where it is largest and smallest. (*hint: it is easier to work with the square of the speed as you might have observed while doing the HW problems from sec 11.4*)

2. Annie the ant likes extreme sports. One day she sees a bicycle wheel approaching and she jumps on as it passes by. She rides on the wheel for 2 full revolutions, then jumps off when she is at ground level. The path Annie takes as she rides on the wheel is described by $\vec{r}(t) = (t - \sin t)\hat{\imath} + (1 - \cos t)\hat{\jmath}$, where $\hat{\imath}$ is parallel to the ground surface.
   (a) How long does Annie's ride last ?
   (b) How far does Annie travel while on the wheel ? (***Exam 1, Spring 09***)

3. Find the **curvature** and **torsion** of the curve $x = a\cos t$, $y = a\sin t$, $z = bt$; where a, b are constants.

4. Find $\vec{T}$, $\vec{N}$, $\vec{B}$, κ and τ for the plane curve $\vec{r}(t) = (2t + 3)\hat{\imath} + (5 - t^2)\hat{\jmath}$.

Well, so much for motivation . Good luck on your test ! ☺