Dated: 31^{st} *March, 2010*

1 Instructions

1. Turn in your solution to any **one** of the following questions.
2. The means to solving the problem is more important than the end result. Infact, while evaluating the solutions to this take home quiz I will not care much about the numerical answer that you provide. Correct mathematical (conceptual) arguments may fetch you all the alloted points.
3. You may work in groups and talk to anybody you like about these problems as long as you dont get it solved by them. Mention the name of your collaborators when you turn in your work.
4. The solutions are **due in recitation on April** 29^{th}, 2010**.**
5. Total points = 40. Show all steps.

2 Questions

1. **Introduction to Vectors and Vector Valued functions**

   (a) A ladybug is climbing on a Volkswagen Bug (:= VW). In its starting position, the surface of the VW is represented by the unit semi-circle $x^2 + y^2 = 1, y \geq 0$ in the xy-plane. The road is represented as the x-axis. At time $t = 0$ the ladybug starts at the front bumper, $(1, 0)$ and walks counter-clockwise around the VW at unit speed relative to the VW. At the same time the VW moves to the right at the speed of 10 units/time.

   i. Find the parametric formula for the trajectory of the ladybug, and find it's position when it reaches the rear bumper. (At $t = 0$, the rear bumper is at $(-1, 0)$.)

   ii. Compute the speed of the bug and find where it is largest and smallest.

   (b) Two F-18 Hornets, F_1 and F_2, are travelling together. At time $t = 0$ hours, they separate and follow different paths given by

   $$F_1 : \quad \frac{x-6}{40} = \frac{y+3}{10} = \frac{z+3}{2}$$

   and

   $$F_2 : \quad \frac{x-6}{110} = \frac{y+3}{4} = \frac{z+3}{1}$$

   (all coordinates measured in miles). Due to system malfunctions, F_2 stops its flight at $(446, 13, 1)$ and, in negligible amount of time, lands at $(446, 13, 0)$. 2 hours later, F_1 is informed of this fact and heads towards F_2 at 175 mph. How long will it take F_1 to reach F_2 and rescue its crew ?

2. **Theory of Space Curves**

   Review Handout 4 or refer your textbook for definitions. You may refer fig. (11.28), p-884 from your textbook to answer (a)-(g).

   (a) Define a *plane* curve and a *twisted* curve.

   (b) The following is the equation of a *circular helix*

   $$\vec{x}(t) = (r\cos t, r\sin t, ct), c \neq 0 \tag{1}$$

   Is this a plane curve or a twisted curve ? Why ?

   (c) What is the geometry of the orthogonal projection of (1) on the xy plane ? What is the geometry of the orthogonal projection of (1) on the yz plane ? What is the geometry of the orthogonal projection of (1) on the xz plane ?

   (d) What is the *curvature* of the circular helix, (1) ? What happens as $c \to 0$? Also give an explanation based on your definition of a plane curve and a twisted curve, above.

   (e) Find a representation of the position vector of the *centre of curvature* of any curve $\vec{x}(t)$ in terms of $\vec{x}(t)$ and its derivatives. Now find the *locus* of the centres of curvature of the circular helix, (1).

(f) Now provide a few geometrical arguments to ascertain that the principal normals to a circular helix, (1) intersect the axis of rotation of the cylinder on which (1) lies at *right angles*. (no need to provide a rigorous proof !)

(g) **(Bertrand Curves)** Recall that a curve is a *circular helix* if both of $\kappa(s) \neq 0$ and $\tau(s)$ are constant. On the other hand, a curve $\gamma : I \rightarrow R^3$ with $\kappa(s) \neq 0$ is called a *Bertrand curve* if there exists a curve $\bar{\gamma} : I \rightarrow R^3$ such that the principal normal lines of γ and $\bar{\gamma}$ at $s \in I$ are equal (same). In such a case $\bar{\gamma}$ is called a *Bertrand mate* of γ. For eg. a set of concentric circles belong to the Bertrand family of curves because they share the same principal normals (i.e. the radial lines are the same). **Give an example of a twisted curve which is also a Bertrand curve ?** (This is much easier than you may initially think as the whole point of this entire question was framed in order to help you answer this question !)

3. **Linearization**

Consider the inverted pendulum on a cart as shown in figure(1). Use the ***linearization*** technique and solve some basic kinematics equations from physics to show by how much θ changes with change in force, F ? Length of pendulum = l, mass of pendulum = m_2, mass of cart = m_1.

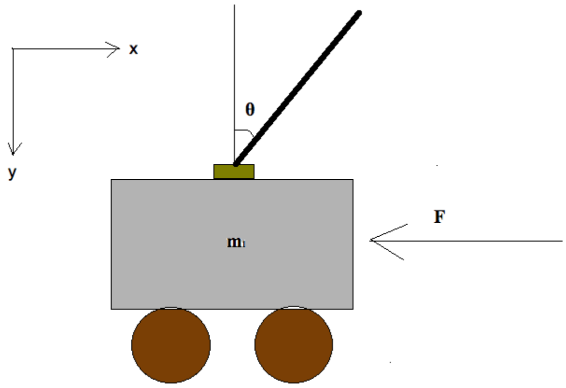

Figure 1: Inverted Pendulum on a cart

4. **Stoke's Theorem and its application**

(a) State Stoke's theorem.

(b) The electromotive force, $\mathcal{E}$ in a circuit C is equal to the *circulation* of the electric field, $\vec{E}$ around the circuit:

$$\mathcal{E} = \oint_C \vec{E} \cdot \hat{t} ds \tag{2}$$

Faraday discovered that in a stationary circuit an electromotive force is induced by a changing magnetic flux. That is,

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{3}$$

where

$$\Phi = \int \int_S \vec{B} \cdot \hat{n} dS \tag{4}$$

t is time (dont confuse it with the tangent vector $\hat{t}$), and S is any capping surface of C. Use this information and the Stokes' theorem to derive the equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5}$$

which is called the *Faraday's law of induction* and is also one of the famous *Maxwell's* equations !