Pollution Mixing and Filtering

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1 Introduction

The mixing and filtering of pollutants in a fluid is an important modeling problem. We consider a tank of water into which pollutant flows at a steady rate while water from the tank flows out at the same rate. Linear differential equations can be used to model this process. We begin by considering a continuous inflow of pollutant and show both graphically and theoretically that the long-term limit of pollutant level depends only on the rate of pollutant inflow. We then adapt our model to include filtering effects. The introduction of perfect and partial filters is considered; it will be shown that the long-term limit of pollutant level still depends on the inflow rate, but it now also depends on the filter efficiency.

2 Model

We wish to consider a tank containing 10 gal of water, initially mixed with 10 lb of pollutant. Water polluted at a constant concentration of c_0 lb/gal flows into the tank at a rate of 1 gal/min. The water in the tank is kept well mixed and is drained at a rate of 1 gal/min. Hence, the volume of water in the tank remains constant (10 gal).

To build a mathematical model of the mixing process, we let $y(t)$ represent the amount of pollutant, measured in lbs, in the tank at time t, measured in minutes. We may now construct a differential equation for y since the rate of change of y is given by the difference of the rates of flow of pollutant in and out of the tank. Pollutant is flowing in at a rate of $(c_0 \text{ lb/gal}) \cdot (1 \text{ gal/min}) = c_0$ lb/min. Since it is well mixed through the water in the tank, it flows out at a rate of $(c(t) \text{ lb/gal}) \cdot (1 \text{ gal/min}) = c(t)$ lb/min, where $c(t)$ is the concentration of pollutant in the mixture at time t. Concentration is the ratio of amount to volume, so

$c(t) = y(t)/10$ (since the volume is held constant at 10 gal). Hence we can model the amount of pollutant by the linear ordinary differential equation (ODE):

$$\begin{aligned} \frac{dy}{dt} &= c_0 - \frac{y}{10} \\ y(0) &= 10. \end{aligned} \tag{1}$$

3 Numerical Investigation

To get a feeling for how the solutions of (1) behave, we use a numerical method to plot approximate solutions for $0 \leq t \leq 40$. Figure 1 shows the long-term behavior of solutions

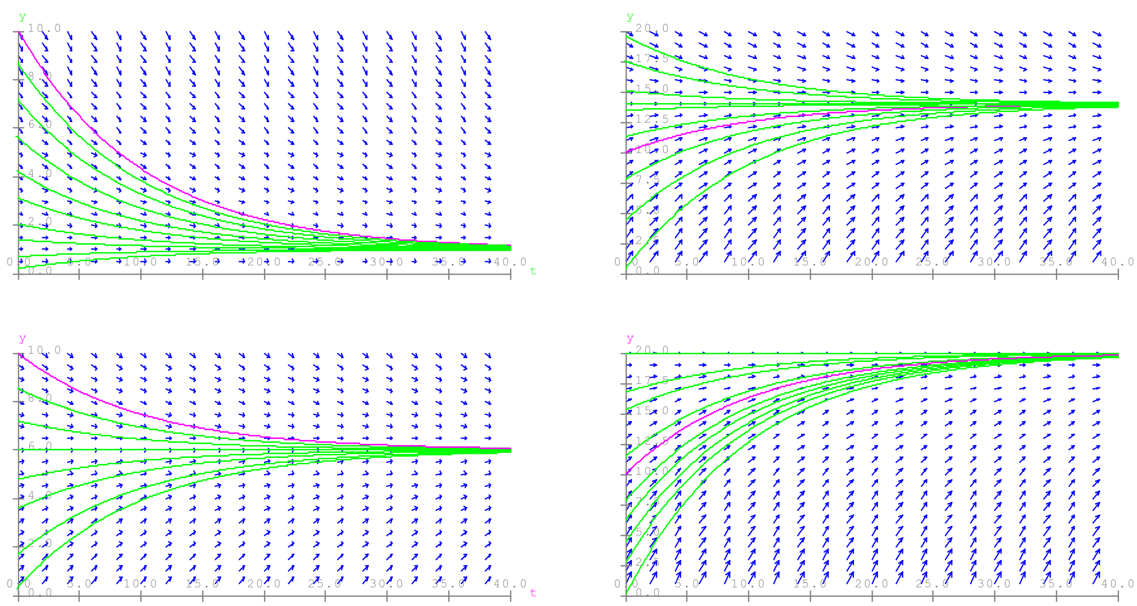

Figure 1: Solution curves for $c_0 = 0.1$ (upper left), $c_0 = 0.6$ (lower left), $c_0 = 1.4$ (upper right) and $c_0 = 2.0$ (lower right). Solution with $y(0) = 10$ is shown in purple; solutions with other initial conditions are shown in green.

for various values of c_0 from 0.1 to 2. We may make two important observations regarding these graphs:

1. Although we are considering the solution with $y(0) = 10$, **all** solutions tend to the same limit as $t \to \infty$.

2. In each case, $y(t) \to 10c_0$ as $t \to \infty$.

It is straightforward to verify (by direct substitution) that $y(t) = 10c_0$ is an equilibrium solution. However, in order to prove the above two observations, we must use the theory of linear differential equations.

4 Linear Theory

Since (1) is a linear ODE, we can use an integrating factor to find $y(t)$ explicitly (see Appendix A):

$$y(t) = 10c_0 + Ke^{\frac{-t}{10}} \tag{2}$$

where K is a constant determined by the initial conditions. For our initial condition $y(0) = 10$,

$$y(t) = 10[c_0 + (1 - c_0)e^{\frac{-t}{10}}]. \tag{3}$$

From this it is clear that, as $t \to \infty$, $y(t) \to 10c_0$ since $e^{\frac{-t}{10}} \to 0$. Indeed, from the general solution (2) it follows that $y(t) \to 10c_0$ regardless of the choice of initial conditions. This, then, proves both of our observations from the numerical solutions.

By following the solution procedure (Appendix A), it becomes clear that the factor of 10 in the long-term limit is a direct result of the volume of our tank — in general we would find that a constant volume of V would lead to a solution which approaches Vc_0 as $t \to \infty$. At first it may appear that it would, therefore, be best to use as small a tank as possible; this leads to an optimal tank volume of 0, which is rather impractical! The confusion is avoided, however, by recalling that $y(t)$ represents the total **amount** of pollutant, so naturally a larger tank will hold more. If we look at the **concentration** of pollutant, given by $c(t) = \frac{y(t)}{V}$, then we find that $c(t) \to c_0$ as $t \to \infty$. This seems entirely reasonable since it means, in physical terms, that the concentration of pollutant in the tank rises or falls to match that of the incoming mixture.

5 Perfect Filtering

We now wish to consider the effects of filtering on the pollutant level in the tank. If a filter is placed in the inflow pipe, the rate of flow of pollutant into the tank (c_0) will be lowered. Furthermore, if this filter is added or removed or if its effectiveness can vary with time, then

c_0 will no longer be a constant. We therefore consider a new model:

$$\frac{dy}{dt} = c_0(t) - \frac{y}{10}. \quad (4)$$

This is still a linear ODE, but $c_0(t)$ is now an arbitrary function of time.

Linear theory still allows us to solve this problem, but a graphical approach may well provide more physical understanding. For this same reason, we will consider a particular example $c(t)$:

$$c(t) = \begin{cases} 1 \text{ for } 0 \leq t \leq 10 \\ 0 \text{ for } 10 < t \end{cases}$$

This represents a constant inflow of 1 lb/min for the first 10 minutes, after which time a perfect filter is added to the pipe. Figure 2 shows the numerical solution of (4) with the

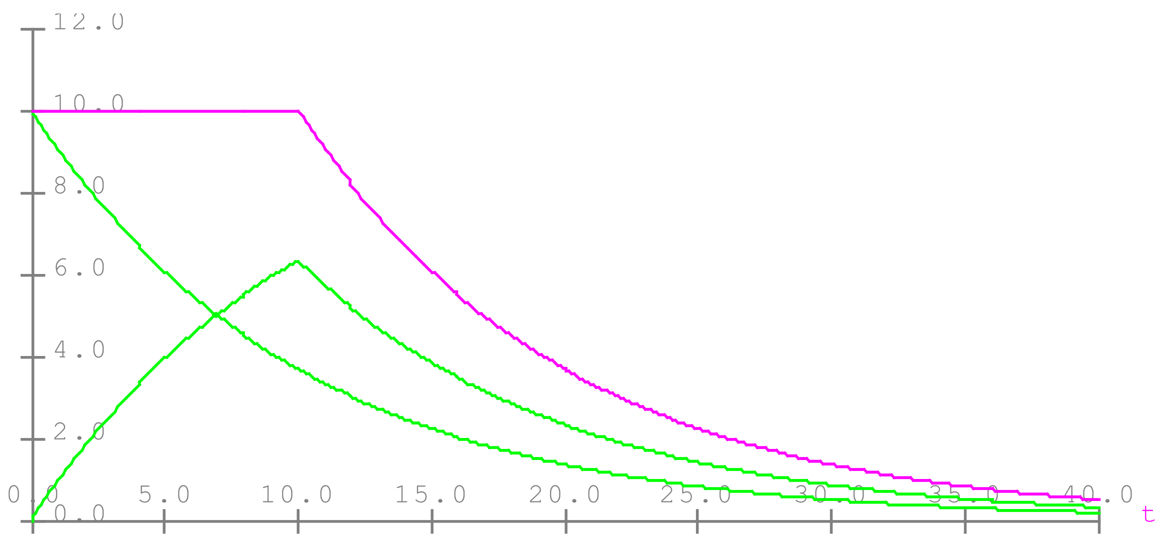

Figure 2: Effect of perfect filter. Full solution shown in purple; green curves represent components of solution due to responses to input and initial conditions.

given $c(t)$ and $y(0) = 10$. Since we start at the equilibrium value (*i.e.* same concentration in the tank as the inflow), the amount of pollutant stays constant until the filter is added. At this point, the amount decreases exponentially towards 0. This is exactly what we would expect physically, since the pollutant will be removed in the outflowing mixture and replaced by pure water from the inflow.

Figure 2 also shows how the solution can be separated into two parts: a response to inflow and a response to initial conditions. This is possible because (4) is linear. Suppose that y_1 is a solution of (4) with initial condition $y(0) = 0$; we say that y_1 is the response to input (inflow, in this case). Also suppose that y_2 is a solution of

$$\frac{dy}{dt} = -\frac{y}{10}$$

with initial condition $y(0) = 10$; we say that y_2 is the response to initial conditions. We can now see that

$$\begin{aligned} \frac{d[y_1 + y_2]}{dt} &= \frac{dy_1}{dt} + \frac{dy_2}{dt} \\ &= c_0(t) - \frac{y_1}{10} - \frac{y_2}{10} \\ &= c_0(t) - \frac{[y_1 + y_2]}{10} \end{aligned}$$

i.e. $[y_1 + y_2]$ is a solution of (4) with, clearly, $[y_1 + y_2](0) = y_1(0) + y_2(0) = 10$. This demonstrates that the two parts, y_1 and y_2 do indeed constitute the full solution $y(t)$.

From Figure 2 it can be seen that the response to initial conditions decays exponentially, while the response to inflow initially increases, but then also decays once the filter is in place.

6 Partial Filtering

The effect of perfect filtering was predictable and physically irrelevant. Of more interest is the response of our system to a more realistic filter which we model by

$$c(t) = \begin{cases} 1 \text{ for } 0 \le t \le 10 \\ \frac{1}{2} \text{ for } 10 < t \end{cases}$$

The amount of pollutant is still modeled using (4). Again we start with $y(0) = 10$ and use a numerical method to find the solution and its components. Figure 3 shows the response to this 50% efficient filter. Again the amount of pollutant remains at equilibrium until the filter is added. Now, however, the filter does not remove all of the pollutant. Although it is difficult to see on this time-scale, the solution does, in fact, tend to 5 lb as $t \to \infty$. Note that this is precisely the value that would be reached given a steady inflow of 0.5 lb/min. Indeed, since the response to initial conditions again decays exponentially, the long-term behavior is precisely that of the response to inflow. Hence we may apply our findings from the unfiltered regime to this system after the filter is in place.

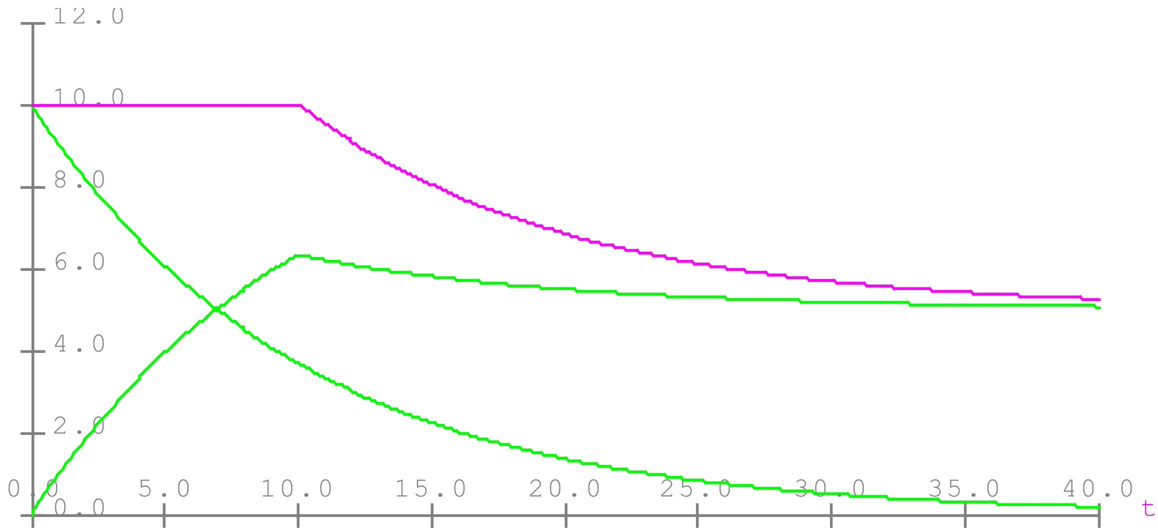

Figure 3: Effect of partial filter. Full solution shown in purple; green curves represent components of solution due to responses to input and initial conditions.

These observations allow us to draw an important conclusion: the long-term limit of the pollutant level depends only on the effective inflow rate — *i.e.* the inflow rate of pollutant not stopped by the filter. Alternatively, the long-term pollutant level is determined by the inflow rate and the efficiency of the filter. Note that these findings apply only to the case where a filter is put in place and then left for a long period of time.

7 Conclusion

A constant-volume fluid mixing situation was successfully modeled using linear ODEs. Although partial filtering was considered, the model may be improved by considering a continuously varying effective inflow, in order to simulate the decreasing efficiency of a real filter.

It was shown both graphically and theoretically that, if the mixing tank is given a steady inflow for a long period of time, the amount of pollutant in the tank will tend toward an equilibrium value that depends only on the effective inflow rate. Moreover, the concentration of pollutant will tend to the same effective concentration as the inflow. It was specifically demonstrated that there is no long-term effect due to the initial amount of pollutant.

A Solution of (1)

In order to solve (1), we first write it in normal linear form:

$$\frac{dy}{dt} + \frac{1}{10}y = c_0.$$

We can now solve this by multiplying through by the integrating factor $e^{\frac{t}{10}}$:

$$\begin{aligned}
& e^{\frac{t}{10}}\left(\frac{dy}{dt} + \frac{1}{10}y\right) = c_0 e^{\frac{t}{10}} \\
\Rightarrow\quad & \frac{d}{dt}\left(e^{\frac{t}{10}}.y\right) = c_0 e^{\frac{t}{10}} \\
\Rightarrow\quad & e^{\frac{t}{10}}.y = \int_0^t c_0 e^{\frac{s}{10}}\,ds \\
\Rightarrow\quad & e^{\frac{t}{10}}.y = 10c_0 e^{\frac{t}{10}} + K \\
\Rightarrow\quad & y(t) = 10c_0 + Ke^{\frac{-t}{10}}.
\end{aligned}$$

Now applying $y(0) = 10$:

$$\begin{aligned}
& y(0) = 10c_0 + K = 10 \\
\Rightarrow\quad & K = 10 - 10c_0 = 10(1 - c_0).
\end{aligned}$$

Hence,

$$y(t) = 10\left[c_0 + (1 - c_0)e^{\frac{-t}{10}}\right].$$