1. (16 points) Let $f(x,y) = \frac{x\sqrt{1-(x^2+y^2)}}{x-y}$.

   (a) What is the domain of $f(x,y)$? Sketch it.

   (b) Describe the domain of f in terms of open/closedness and boundedness.

   (c) Find the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along the x-axis.

   (d) Find the limit of $f(x,y)$ as $(x,y) \to (0,0)$ along the y-axis.

   (e) What, if anything, do your answers to (c) and (d) tell you about $\lim_{(x,y)\to(0,0)} f(x,y)$?

   (f) Is $f(x,y)$ continuous at $(0,0)$?

   Solution:

   (a) $D = \{(x,y) : x^2+y^2 \le 1, x \ne y\}$ (looks like unit disk with line $y = x$ removed).

   (b) D is neither open nor closed because it contains some but not all of its boundary points. It is bounded because it fits inside a circle of finite radius.

   (c) On the x-axis $y = 0$, so $f(x,y) = f(x,0) = \frac{x\sqrt{1-x^2}}{x}$. Hence the limit is $\lim_{x\to 0}\sqrt{1-x^2} = 1$.

   (d) On the y-axis $x = 0$, so $f(x,y) = f(0,y) = \frac{0\sqrt{1-y^2}}{-y}$. Hence the limit is $\lim_{y\to 0} 0 = 0$.

   (e) Since the limits from different directions are different, the limit does not exist.

   (f) No, f is not continuous since neither the limit nor $f(0,0)$ exist (so, clearly, they can't be equal).

2. (18 points) Let P_1 be the plane $2x - y - 2z = 4$ and P_2 be the plane $3x + 2y + 2z = -8$.

   (a) Find the angle between P_1 and P_2.

   (b) Find parametric equations for the line of intersection of P_1 and P_2.

   (c) Find the distance from the point $A(1,-4,-1)$ to P_1.

   Solution:

   (a) The normals of the planes are $\mathbf{n}_1 = 2\hat{\imath} - \hat{\jmath} - 2\hat{k}$ and $\mathbf{n}_2 = 3\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$. Hence $\mathbf{n}_1 \cdot \mathbf{n}_2 = (2)(3) + (-1)(2) + (-2)(2) = 6 - 2 - 4 = 0$. So the planes are orthogonal (angle between them is $\pi/2$).

   (b) Find a point on both planes. Set (for example) $z = 0$ and solve simultaneously for x and y to get $(0,-4,0)$. The direction of the line is given by

   $$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & -2 \\ 3 & 2 & 2 \end{vmatrix} = (-2+4)\hat{\imath} - (4+6)\hat{\jmath} + (4+3)\hat{k} = 2\hat{\imath} - 10\hat{\jmath} + 7\hat{k}$$

   Hence the equation for the line is $x = 2t, y = -4 - 10t, z = 7t$.

   (c) Let S be some point on P_1. For example, $S(2,0,0)$. Then $\vec{AS} = \hat{\imath} + 4\hat{\jmath} + \hat{k}$, so $\vec{AS} \cdot \mathbf{n}_1 = 2 - 4 - 2 = -4$. The magnitude of $\mathbf{n}_1$ is $|\mathbf{n}_1| = \sqrt{4+1+4} = \sqrt{9} = 3$. Using the formula: $d = |\frac{-4}{3}| = 4/3$.

3. (16 points) Consider a flat square plate $\{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$. The temperature on the plate is given by $T(x, y) = x^2y$.

   (a) If an ant walks across the plate on some path $(x(t), y(t))$, what (symbolically) is $\frac{dT}{dt}$?

   (b) Suppose the ant follows the parabolic path $(x(t), y(t)) = (t/2, t^2/4)$. Sketch the plate and the ant's path from $t = 0$ to $t = 2$.

   (c) If the ant follows the parabolic path in (b), what is $\frac{dT}{dt}$ when the ant is at the point $(x, y) = (1/2, 1/4)$?

   (d) Suppose the plate is cooling, so that $T(x, y, t) = x^2ye^{-t}$. Now what is $\frac{dT}{dt}$ (for a general path $(x(t), y(t))$)?

   Solution:

   (a) $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = 2xy\frac{dx}{dt} + x^2\frac{dy}{dt}$

   (b) Looks like a square with parabolic curve from origin to top-right corner $(1, 1)$.

   (c) From (a), $\frac{dT}{dt} = \frac{dy}{dt} = 2xy\frac{dx}{dt} + x^2\frac{dy}{dt} = 2(t/2)(t^2/4)(1/2) + (t/2)^2(2t/4) = t^3/8 + t^3/8 = t^3/4$. The ant is at $(1/2, 1/4)$ at $t = 1$. So $\frac{dT}{dt} = 1/4$.

   (d) Now $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial t} = 2xye^{-t}\frac{dx}{dt} + x^2e^{-t}\frac{dy}{dt} - x^2ye^{-t}$.

4. (16 points) Consider $\mathbf{r} = 3\hat{\imath} + 2\sin(2t)\hat{\jmath} + 2\cos(2t)\hat{k}$

   (a) Find the unit tangent to the curve.

   (b) Find the unit normal to the curve.

   (c) Find the curvature of the curve, as a function of t.

   Solution:

   (a) $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$; $\mathbf{v} = (\frac{d}{dt}3)\hat{\imath} + (\frac{d}{dt}2\sin(2t))\hat{\jmath} + (\frac{d}{dt}2\cos(2t))\hat{k} = 0\hat{\imath} + 4\cos(2t)\hat{\jmath} - 4\sin(2t)\hat{k}$. Hence $|\mathbf{v}| = \sqrt{16\sin^2(2t) + 16\cos^2(2t)} = 4$, and so $\mathbf{T} = \cos(2t)\hat{\jmath} - \sin(2t)\hat{k}$.

   (b) $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$; $\frac{d\mathbf{T}}{dt} = -2\sin(2t)\hat{\jmath} - 2\cos(2t) \Rightarrow |\frac{d\mathbf{T}}{dt} = 2$, so $\mathbf{N} = -\sin(2t)\hat{\jmath} - \cos(2t)\hat{k}$.

   (c) $\kappa = |\frac{d\mathbf{T}}{ds}| = |\frac{d\mathbf{T}}{dt}\frac{ds}{dt}| = |\frac{d\mathbf{T}}{dt}|\frac{1}{|\mathbf{v}|}$. From above, $\kappa = (2)(1/4) = 1/2$ (for all time).

— Over —

5. (18 points) Consider the function $p(w,x,y,z) = w^2xy - e^{wyz}$.
   (a) Find the derivatives $\frac{\partial p}{\partial w}, \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}, \frac{\partial^2 p}{\partial x \partial z}, \frac{\partial^4 p}{\partial w^2 \partial x \partial y}$.
   (b) Which variable(s) is p most sensitive to at the point $(-2,3,1,0)$?
   (c) Give the equation for the linearization of p at $(-2,3,1,0)$.

   Solution:

   (a) $p_w = 2wxy - yze^{wyz}$, $p_x = w^2y$, $p_y = w^2x - wze^{wyz}$, $p_z = -wye^{wyz}$, $p_{xz} = \partial_z(p_x) = 0$, $p_{wwxy} = \partial_w^2(\partial_x(p_y)) = \partial_w^2(w^2) = \partial_w(2w) = 2$.
   (b) Evaluating the first derivatives at $(-2,3,1,0)$: $p_w = (2)(-2)(3)(1) - 0 = -12$, $p_x = (-2)^2(1) = 4$, $p_y = (-2)^2(3) - 0 = 12$, $p_z = (2)(1)e^0 = 2$. So p is most sensitive to w and y.
   (c) $p(-2,3,1,0) = (-2)^2(3)(1) - e^0 = 11$, so $L(w,x,y,z) = 11 + (-12)(w+2) + (4)(x-3) + (12)(y-1) + (2)(z-0) = 11 - 12w - 24 + 4x - 12 + 12y - 12 + 2z = -37 - 12w + 4x + 12y + 2z$.

6. (16 points) Match each of the pictures shown (a)-(d) with one of the equations below. (Note: there are more equations than pictures, so four equations will be unused.) No work need be shown for this problem.
   (a) 4
   (b) 8
   (c) 3
   (d) 7

— Useful and interesting formulae —

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\vec{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\vec{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \qquad \text{where} \qquad a_T = \frac{d}{dt}|\mathbf{v}|, \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

$$|E(x,y)| \le \frac{M}{2}\left(|x-x_0| + |y-y_0|\right)^2 \qquad \text{where} \qquad |f_{xx}|, |f_{xy}|, |f_{yy}| \le M$$