
INSTRUCTIONS: Answer all questions completely, carefully and thoroughly. Answers must be neat and legible (*i.e.* what you hand in must be a final copy, not something thrown together at the last moment and still in progress). Show working and reasoning. You may use any resources available to you, but the final work handed in must be yours alone; the instructors and TAs will not answer questions directly related to these problems. Integrals may be evaluated using the tables in the back of your textbook, but these formulae must be referenced if used. You may ask other instructors, but you must show them this exam sheet, so that they know they can help you within reason. You must cite any reference for any result not derived yourself. **Due: 4pm, Friday July 22** — drop off in Applied Math office, ECOT 225

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1. (20 points)

   (a) Consider a solid sphere of radius a, centered about the point $(0, 0, z_0)$. If the sphere has constant density δ, show that $M_{xy} = \frac{4}{3}\pi a^3 z_0 \delta$. Show that this makes sense by calculating $\bar{z}$ [no actual calculus required for this part].

   (b) Consider the cone $z = 3\sqrt{x^2 + y^2}$ from $z = 0$ to $z = h$. The mass and moment formulae for surfaces are very similar to those for wires, thin plates and solid objects; they are given on page 1103 of the textbook. If the cone has constant density δ, show that $M_{xy} = \frac{2\sqrt{10}}{27}\pi h^3 \delta$, and $M = \frac{\sqrt{10}}{9}\pi h^2 \delta$.

   (c) *Ryan & Matt's Olde Time Ice Cream Shoppe* sells old-fashioned ice-cream cones made from a cone with one or two spheres of ice cream on top. Use geometry to show that the center of the lower sphere is at $(0, 0, \sqrt{10}a)$ (taking $(0, 0, 0)$ to be the base of the cone), where a is the radius of the sphere, assuming that the sphere of ice cream rests on the sides of the cone so that the surfaces are tangent at that point.

   (d) Use the result in (c) to determine where the centers of the two spheres of ice cream would be if the cone has a height $h = 3$ and the spheres have radius $a = 1$.

   (e) Use the results in (a)–(d) to determine the centers of mass for one- and two-scoop ice creams, given that the density of the ice cream is $\delta = 0.5$ and the density of the cone is $\delta = 0.1$. [You may use symmetry arguments, where appropriate.]

   (f) The owners of *Ryan & Matt's Olde Time Ice Cream Shoppe* (*Giganticorp Dairy Industries International, Inc.*) want you to design those plastic tray/stand thingies that hold an ice-cream cone upright. Ideally, the radius of the circle would be large enough so that the center of mass is below the level of the holder when the ice cream is resting in the holder. Why? Is this possible? If not, how would you design the holder? What factors would you consider, and what (if any) calculus would you use to help you?

2. (10 points) The air in the atmosphere is said to be in geostrophic balance if $\hat{k} \times \mathbf{u} = -\frac{1}{f\rho}\nabla p$, where $\mathbf{u}$ is the wind velocity, p is the pressure, f is the "Coriolis parameter" (constant) and ρ is the air density. This equation is a good approximation under certain conditions, including: the density can be assumed to be constant, and the wind velocity field is two dimensional, so $\mathbf{u} = u(x,y)\hat{\imath} + v(x,y)\hat{\jmath}$ (and $p = p(x,y)$).
   (a) Show that $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{A}) = 0$ for any vectors $\mathbf{A}, \mathbf{B}$.
   (b) Use the fact in (a) to show that, if the atmosphere is in geostrophic balance, the wind velocity ($\mathbf{u}$) is always parallel to the isobars (lines of constant pressure). [Hint: isobars are contours of the pressure function.]

3. (10 points) It is well-known that Helen of Troy had a "face that launched a thousand ships". We can therefore define the milliHelen to be the unit of ship-launching potential based on the amount of face required to launch one ship (1 milliHelen = 1 ship/in^2 of face). Helen's sister Nellie has a face that can be approximated as an ellipse $x^2 + y^2/4 \leq 1$ with a ship-launching density function of $s(x,y) = 200 - 2(x^2/4 + y^2 - y) + 2e^{-2(x^2+y^2/4)}$ (in milliHelens).
   (a) Use *Mathematica* to plot the contours of s over the region $-1 \leq x \leq 1, -2 \leq y \leq 2$, as well as the boundary of the ellipse (on the same plot). Show on the plot where the maximum and minimum of s would be (be sure to justify your choices).
   (b) If Leipzig (Paris's brother) abducts Nellie, how many ships will be launched? [Hint: the transformation $x = r\cos(\theta), y = 2r\sin(\theta)$ will be useful.]

4. (10 points) The streamlines of a planar fluid flow are the smooth curves traced by the fluid's individual particles. The vector field $\mathbf{u} = u(x,y)\hat{\imath} + v(x.y)\hat{\jmath}$ gives the tangent vectors of the streamlines. Suppose $\nabla \cdot \mathbf{u} \neq 0$ throughout a simply connected region R. Show that the flow (in R) cannot have any streamlines that are closed loops. [Hint: try a "proof by contradiction": take $\nabla \cdot \mathbf{u} \neq 0$, as given, but assume that there is a closed loop streamline; show that this leads to a contradiction, and so conclude that, in fact, there can't be any closed loops.]

5. (all points) Sign and date the honor code (printed on the next page) and attach it to your work.

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

Signature: ____ Date: _____________