
INSTRUCTIONS: Computers, calculators, books, notes, flying monkeys, *etc.* are not permitted. Some (possibly) useful formulae are attached. Write your name, your instructor's name, and the color of your exam sheet on the front of your bluebook. Work all problems. Start each problem on a **new page**. Show your work clearly and $\boxed{\text{box}}$ your final answer. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

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1. (16 points) Multi-choice — choose the best answer. No work need be shown for this question.

   (a) If $\dfrac{\partial}{\partial s}\left(\dfrac{\partial g}{\partial t}\right) = \dfrac{t^2\cos(st)}{s^2+6}$, then $\dfrac{\partial}{\partial t}\left(\dfrac{\partial g}{\partial s}\right) = \dfrac{t^2\cos(st)}{s^2+6}$.

   A. True.
   B. True if $\frac{\partial g}{\partial s} = \frac{\partial g}{\partial t}$.
   C. True except, possibly, at $(s,t) = (0,0)$.
   D. True except, possibly, at $(s,t) = (0,0)$, if g is continuous everywhere.
   E. False.

   (b) If $f(x,y)$ is continuous everywhere, $\int_0^2 \int_0^{y^2} f(x,y)\,dx\,dy =$

   A. $\int_0^{y^2} \int_0^2 f(x,y)\,dy\,dx$
   B. $\int_0^2 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx$
   C. $\int_0^4 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx$
   D. $\int_0^4 \int_{\sqrt{x}}^2 f(x,y)\,dy\,dx$
   E. None of the above

   (c) If $p = p(a,b,c,t)$ and $a = a(x,y)$, $b = b(x,t)$, $c = c(y,t)$, then

   A. $\dfrac{\partial p}{\partial t} = \dfrac{\partial p}{\partial b}\dfrac{\partial b}{\partial t} + \dfrac{\partial p}{\partial c}\dfrac{\partial c}{\partial t}$.
   B. $\dfrac{\partial p}{\partial t} = \dfrac{dp}{db}\dfrac{\partial b}{\partial t} + \dfrac{dp}{dc}\dfrac{\partial c}{\partial t} + \dfrac{dp}{dt}$.
   C. $\dfrac{\partial p}{\partial t} = \dfrac{\partial p}{\partial a}\dfrac{\partial a}{\partial t} + \dfrac{\partial p}{\partial b}\dfrac{\partial b}{\partial t} + \dfrac{\partial p}{\partial c}\dfrac{\partial c}{\partial t}$.
   D. $\dfrac{dp}{dt} = \dfrac{\partial p}{\partial b}\dfrac{db}{dt} + \dfrac{\partial p}{\partial c}\dfrac{dc}{dt} + \dfrac{\partial p}{\partial t}$.
   E. None of the above.

   (d) If $f_{xy}(a,b) < 0$ and $f_{yy}(a,b) < 0$ then $f(x,y)$

   A. has a minimum at (a,b).
   B. has either a maximum or a minimum at (a,b).
   C. has either a maximum or a saddle at (a,b).
   D. has a saddle at (a,b).
   E. None of the above.

2. (30 Points) Consider the function $f(x,y) = e^{x^2+2y^2}$.

   (a) Find and classify all critical points of $f(x,y)$.

   (b) Now, find the global min/max values of $f(x,y)$ over the region $R = \{(x,y) : 2x^2 + 2xy + 5y^2 \le 9\}$.

3. (20 Points) The gravitational potential energy of a body of mass m in the presence of a mass M is given by $P(x,y,z) = \dfrac{-GMm}{\sqrt{x^2+y^2+z^2}}$, where the origin is at the center of the mass M (and G is the universal gravitational constant).

   (a) Describe the level sets of P geometrically. Be sure to explain how the level sets change as the value of P is changed. Draw a sketch to illustrate your answer.

   (b) Find a unit normal vector to the level set where $P(x,y,z) = P_0$ (for a general (x,y,z) point on this set).

   (c) The mass m is in "free-fall" if it follows a path that decreases P the most rapidly. Using principles of Calc III and your answers to (a) and (b), explain what direction the mass m will go when in free-fall (and why). Describe this direction geometrically and how it relates to the level sets. Explain briefly why this makes sense physically.

4. (14 Points) Astronomers have discovered a new planet in orbit around a distant star. From their observations, they have estimated that the planet's orbital period is $T = 4 \pm 0.05$ years, and its semi-major axis is $a = 2 \pm 0.05$ AU. (The AU is an astronomical unit of length $\approx 1.5 \times 10^{11}$ m.) According to Kepler's third law, $\dfrac{T^2}{a^3} = \dfrac{4\pi^2}{GM}$, where G is the universal gravitational constant and M is the mass of the star-planet system. Using this, and the values $T = 4$, $a = 2$, the mass was determined to be 10^{30} kg.

   (a) Estimate the error in the calculated mass; the value of $\dfrac{4\pi^2}{G}$ is $2 \times 10^{30} \frac{\text{kg}\cdot\text{year}^2}{\text{AU}^3}$.

   (b) With the vast quantities of grant money now coming to them, the astronomers have hired a student to analyze the data and improve the error estimates on a and/or T. With a conference deadline approaching, the student has time to work on only one; which measurement, a or T, will give the astronomers the better return, in terms of improved accuracy of M?

5. (20 Points) A remote-control submarine is moving through the ocean along a path $\mathbf{r}(t)$; the temperature of the water is given by a function $T(x,y,z)$. At a given time t^* you know that $\mathbf{r}(t^*) = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{v}(t^*) = \mathbf{j} - 3\mathbf{k}$ and $\mathbf{a}(t^*) = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$; you also know that $T(1,-2,-1) = 5$ and $\nabla T|_{(1,-2,-1)} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

   (a) Calculate the rate of change of the temperature with respect to **time** that the submarine experiences (at time t^*), or explain why it cannot be calculated.

   (b) Calculate the rate of change of the temperature with respect to **distance** that the submarine experiences (at time t^*), or explain why it cannot be calculated.

   (c) Suppose $T = 5$ is too cold for the submarine to function properly, so its controllers want to send a command to change direction. Which direction should they redirect the submarine to so that the temperature increases most rapidly? (You may assume instantaneous communication and redirection.)

   (d) Suppose (ignoring part (c)) that $T = 5$ is a perfect operating temperature for the submarine, so the controllers want to keep it at that temperature. However, seaweed has jammed some of the control surfaces, so the submarine cannot be maneuvered in the $\mathbf{i}$ direction. Can the controllers still move the submarine in a direction that keeps T constant? If so, what direction? If not, why not?

— Useful and interesting formulae —

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\vec{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\vec{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(u)|\,du \qquad \hat{\mathbf{T}} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \hat{\mathbf{N}} = \frac{d\hat{\mathbf{T}}/ds}{|d\hat{\mathbf{T}}/ds|} = \frac{d\hat{\mathbf{T}}/dt}{|d\hat{\mathbf{T}}/dt|} \qquad \mathbf{B} = \hat{\mathbf{T}}\times\hat{\mathbf{N}}$$

$$\kappa = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{[\dot{x}^2+\dot{y}^2]^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\hat{\mathbf{N}}$$

$$\mathbf{a} = a_T\hat{\mathbf{T}} + a_N\hat{\mathbf{N}} \qquad \text{where} \qquad a_T = \frac{d}{dt}|\mathbf{v}|, \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

$$\frac{df}{ds} = D_{\mathbf{u}}f = (\nabla f)\cdot\mathbf{u}$$

$$\text{Discriminant: } \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$$

$$\nabla f = \lambda\nabla g \qquad g = 0$$

$$\begin{aligned} f(x,y) &= f(0,0) + \left(\left.\frac{\partial f}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial f}{\partial y}\right|_{(0,0)} y\right) \\ &+ \frac{1}{2}\left(\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} x^2 + 2\left.\frac{\partial f^2}{\partial x\partial y}\right|_{(0,0)} xy + \left.\frac{\partial f}{\partial y^2}\right|_{(0,0)} y^2\right) \\ &+ \ldots + \frac{1}{n!}\left.\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n\right|_{(0,0)} f + \ldots \end{aligned}$$

$$|E(x,y)| \le \frac{M}{2}\left(|x-x_0| + |y-y_0|\right)^2 \qquad \text{where} \qquad |f_{xx}|, |f_{xy}|, |f_{yy}| \le M$$