
INSTRUCTIONS: Computers, calculators, books, notes, flaming monkeys, *etc.* are not permitted. Some (possibly) useful formulae are attached. Write your name, your instructor's name, your recitation section number, and the name of your favorite Teletubby on the front of your bluebook. Work all problems. Start each problem on a **new page**. Show your work clearly and box your final answer. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

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1. (40 points) Consider the integral $I = \int_0^{2\pi}\int_1^2\int_0^z r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_2^{\sqrt{8}}\int_0^{\sqrt{8-z^2}} r\,dr\,dz\,d\theta.$
   (a) Make a clear sketch of the region of integration. (If you have trouble with this, you may "buy" the answer for 7 points.)
   (b) Express I in cylindrical coordinates using the order $dz\,dr\,d\theta$.
   (c) Express I in spherical coordinates using the order $d\rho\,d\phi\,d\theta$.
   (d) Express I in spherical coordinates using the order $d\phi\,d\rho\,d\theta$.
   (e) Evaluate one of the integrals above to determine the value of I.

2. (25 points) Consider the integral $I = \iint_R 21y\,dx\,dy$ where R is the region above the x-axis and bounded by the curves $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$.
   (a) The transformation $x = u^2 - v^2$, $y = 2uv$, for $u \geq 0$ and $v \geq 0$, results in a region of integration R' in the uv-plane bounded by $u = 0$, $u = 1$, $v = 0$ and $v = 1$. Make two clear sketches, one of the original region of integration R in the xy-plane, and one of the region R' in the uv-plane. (Be sure to label your axes, boundaries, and so on!)
   (b) Rewrite the integral over the region R' in the uv-plane.
   (c) Evaluate I in terms of u and v.
   (d) ~~Prove~~ Show that the transformation $x = u^2 - v^2$ and $y = 2uv$, for $u \geq 0$ and $v \geq 0$, results in the region of integration R' described in part (a).

3. (15 points) While a skinny worm of cross-sectional area A eats its way through a really, really big apple, its head travels along the path defined by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$. The density of the apple (mass per unit volume) is given by $\rho(x, y, z) = \dfrac{2 + xyz}{\sqrt{1 + 4y + 9xz}}$. Approximately how much of the apple's mass has the worm eaten for $0 \leq t \leq 1$?

4. (20 points) Consider an object moving around in the xy-plane in a force field described by $\mathbf{F} = x^2\,\mathbf{i} - xy\,\mathbf{j}$.
   (a) At first, the object moves along quarter-circle path in a counterclockwise direction from $(R, 0)$ to $(0, R)$. Calculate the **work** done while moving along this portion of the path.
   (b) Then the object moves in a straight line from $(0, R)$ to the location $(-R, 0)$. Calculate the **flux** while moving along this second portion of the path.
   (c) If the object accelerates as it moves along the path described in part (a), is more, less, or the same amount of work done. Justify your answer.

— Useful and interesting formulae —

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\vec{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\vec{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(u)|\,du \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{[\dot{x}^2+\dot{y}^2]^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \qquad \text{where} \qquad a_T = \frac{d}{dt}|\mathbf{v}|, \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

$$\frac{df}{ds} = D_{\mathbf{u}}f = (\nabla f)\cdot\mathbf{u}$$

$$\text{Discriminant: } \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$$

$$\nabla f = \lambda\nabla g \qquad g = 0$$

$$\begin{aligned} f(x,y) &= f(0,0) + \left(\left.\frac{\partial f}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial f}{\partial y}\right|_{(0,0)} y\right) \\ &+ \frac{1}{2}\left(\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} x^2 + 2\left.\frac{\partial f^2}{\partial x\partial y}\right|_{(0,0)} xy + \left.\frac{\partial f}{\partial y^2}\right|_{(0,0)} y^2\right) + \ldots + \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n\bigg|_{(0,0)} f + \ldots \end{aligned}$$

$$|E(x,y)| \le \frac{M}{2}\left(|x-x_0| + |y-y_0|\right)^2 \qquad \text{where} \qquad |f_{xx}|, |f_{xy}|, |f_{yy}| \le M$$

$$M = \iint_R \delta(x,y)\,dA \qquad M_x = \iint_R y\delta(x,y)\,dA \qquad M_y = \iint_R x\delta(x,y)\,dA \qquad (\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

$$I_x = \iint_R y^2\delta(x,y)\,dA \qquad I_y = \iint_R x^2\delta(x,y)\,dA \qquad I_0 = \iint_R (x^2+y^2)\delta(x,y)\,dA$$

$$\begin{aligned} x &= r\cos(\theta) & r &= \rho\sin(\phi) & x &= \rho\sin(\phi)\cos(\theta) \\ y &= r\sin(\theta) & z &= \rho\cos(\phi) & y &= \rho\sin(\phi)\sin(\theta) \\ z &= z & \theta &= \theta & z &= \rho\cos(\phi) \end{aligned}$$

$$\rho = \sqrt{x^2+y^2+z^2} \qquad r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}(y/x) \qquad \phi = \tan^{-1}(r/z)$$

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left\| \begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{array} \right\| du\,dv\,dw$$

$$\begin{aligned} \text{Work: } \int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy \\ \text{Flux: } \int_C \mathbf{F}\cdot\mathbf{n}\,ds &= \int_C M\,dy - N\,dx \end{aligned}$$

$$\text{Conservative field: } \nabla\times\mathbf{F} = \mathbf{0} \Leftrightarrow \begin{cases} \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \\ \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z} \\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \end{cases}$$