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**INSTRUCTIONS:** Computers, calculators, books, notes, flying penguins, *etc.* are not permitted. Some (possibly) useful formulae are attached. Write your name, your instructor's name, and your favorite NHL team on the front of your bluebook. Work all problems. Start each problem on a **new page**. Show your work clearly and box your final answer. A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

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1. (20 points) State whether the following statements are (always) "TRUE" or "FALSE" (not always true), for twice continuously differentiable functions  $f, \mathbf{F}, \mathbf{G}$ . You MUST write the full word TRUE or FALSE — T/F will NOT be graded. You do NOT need to show your working or reasoning.
  - (a)  $\nabla \cdot (\nabla \times \mathbf{F}) = \nabla \times (\nabla f)$  if  $\mathbf{F} = \nabla f$ .
  - (b)  $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\nabla \cdot \mathbf{F})\mathbf{G} + (\nabla \cdot \mathbf{G})\mathbf{F}$ .
  - (c) If  $\mathbf{F} = \nabla \times \mathbf{G}$  then  $\iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, d\sigma = - \iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, d\sigma$  for any  $S_1$  and  $S_2$  that intersect in a closed curve, and have outward normals in opposite directions.
  - (d) If  $S$  is a closed surface and  $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0$  then  $\nabla \cdot \mathbf{F} = 0$  or  $\nabla \times \mathbf{F} = \mathbf{0}$ .
  - (e) If  $\nabla \cdot \mathbf{F} = 0$  and  $\nabla \times \mathbf{F} = \mathbf{0}$  then  $\mathbf{F} = \mathbf{0}$ .
  
2. (40 points) Consider the "shark-fin" cylinder bounded by the surfaces  $x^2 + y^2 = R^2$  and  $x = \frac{y(R-y)}{R}$ , and the planes  $y = 0$ ,  $z = 0$  and  $z = H$ . The temperature distribution in this object is given by  $T = T_0 - xyz$  (where  $T_0$  is a constant). The vector  $\mathbf{F} = -\nabla T$  is called the "heat flux vector".
  - (a) Calculate the heat flux vector  $\mathbf{F} = -\nabla T$ .
  - (b) Sketch the "shark-fin" footprint of the cylinder in the first quadrant of the  $x$ - $y$  plane.
  - (c) The amount of heat crossing any surface can be found by evaluating the surface integral representing the outward flux of  $\mathbf{F}$  over that surface. Set up fully, but do not evaluate, the integral required to calculate the outward heat flux across the top surface  $z = H$ .
  - (d) Show that the flux out of the bottom surface  $z = 0$  exactly cancels that out of the top.
  - (e) What is the outward heat flux across the curved cylindrical surface?
  - (f) What is the net outward heat flux across the remaining two sides ( $x = \frac{y(R-y)}{R}$  and  $y = 0$ )?
  
3. (30 Points) The elliptical island of Suluclac has a coastline given by  $2x^2 + 2xy + 5y^2 = 9$ , The density of penguins\* on the island can be represented by  $p(x, y) = e^{x^2 + 2y^2}$ . You are a well educated, but lost, leopard seal that just finished a Calculus III course and you want to hunt for penguins on the coastline of the island. Find the location(s) on the coast of Suluclac where you should hunt for best results, and the location(s) that would be the least productive.
 

[\*Obviously a special and rare species of tropical penguin, *Sirron elraetii*.]
  
4. (30 Points) Consider an object that travels along a helical path on the surface of the cylinder  $x^2 + y^2 = R^2$ . While starting at the point  $(R, 0, 0)$  and ending at the point  $(R, 0, H)$ , it travels around the cylinder exactly once. Calculate the work done by the particle if it is moving in a force field given by  $\mathbf{F} = (2xy + z^3)\mathbf{i} + (x^2 - \sin y)\mathbf{j} + (3xz^2)\mathbf{k}$ .

5. (20 points) The Celsius temperature in a region of space is given by  $T(x, y, z) = \frac{3}{8}x^2 + yz$ . You are moving on a path described by  $\mathbf{r} = 2t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}$  where time is measured in seconds and distance is measured in meters. Answer the following questions corresponding to the moment you are at position  $P_0$  with coordinates  $(4, 6, 4)$ .
- How fast is the temperature changing in  $^{\circ}\text{C}/\text{sec}$ ?
  - How fast is the temperature changing in  $^{\circ}\text{C}/\text{m}$ ?
  - Estimate the maximum possible decrease in temperature if you could move only 0.1 meters away from  $P_0$ , in any direction.
6. (30 Points) Consider an object bounded on top by the surface  $x^2 + y^2 + (z - R)^2 = R^2$  and the bottom by the surface  $z = \sqrt{x^2 + y^2}$ .
- Find a parametrization of the curve  $C$ , defined by the intersection of the two surfaces.
  - Calculate the circulation of  $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + xz\mathbf{k}$  around  $C$ .
  - If the result in (b) can be verified using a Theorem from Calculus III, state the theorem and set up (but do not evaluate) the appropriate alternate calculation. Otherwise, clearly state “result cannot be verified”.
7. (30 Points) A new paraboloid roof design, described by  $z = x^2 + y^2$ , was proposed by Nick and Tony’s Design Firm for a building exclusively dedicated to the discreet long-term storage and preservation of “spherical cow remains” for the discerning (and ruthless) mathematician. The roof design will be tested on a multi-story, 4-sided building, built straight up from its foundation. The foundation itself is in the first quadrant of the  $xy$ -plane, and each of the four walls is built along one of the curves  $y^2 = x$ ,  $y^2 = 3x$ ,  $x^2 = 2y$ , and  $x^2 = 5y$ . The current design calls for a roofing material density (kg per square meter) given by the function  $\rho = \frac{3xy}{\sqrt{1 + 4(x^2 + y^2)}}$ . Tony, “The Contractor”, has personally asked you to calculate the total mass of the roofing material. Don’t let him down. Tony doesn’t like people who let him down.
- Hint: Nick, “The Nice Guy”, has tipped you off that you are being “tested”. He suggests that at the appropriate point in the calculations you might need the substitution  $u = \frac{y^2}{x}$  and  $v = \frac{x^2}{y}$ , which, he kindly points out, is the same as  $x = u^{1/3}v^{2/3}$  and  $y = u^{2/3}v^{1/3}$ . It’s good for you (and your kneecaps) that Nick took Calc III years ago and still remembers most of it.

— Useful and interesting formulae —

$$\text{proj}_{\mathbf{A}} \mathbf{B} = \left( \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \right) \mathbf{A} \quad d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} \quad d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|$$

$$s(t) = \int_{t_0}^t |\mathbf{v}(u)| du \quad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} \quad \tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N} \quad \text{where} \quad a_T = \frac{d}{dt} |\mathbf{v}|, \quad a_N = \kappa |\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

$$\frac{df}{ds} = D_{\mathbf{u}} f = (\nabla f) \cdot \mathbf{u}$$

$$\text{Discriminant: } \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2$$

$$\nabla f = \lambda \nabla g \quad g = 0$$

$$\begin{aligned} f(x, y) &= f(0, 0) + \left( \frac{\partial f}{\partial x} \Big|_{(0,0)} x + \frac{\partial f}{\partial y} \Big|_{(0,0)} y \right) \\ &+ \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2} \Big|_{(0,0)} x^2 + 2 \frac{\partial^2 f}{\partial x \partial y} \Big|_{(0,0)} xy + \frac{\partial^2 f}{\partial y^2} \Big|_{(0,0)} y^2 \right) \\ &+ \dots + \frac{1}{n!} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^n \Big|_{(0,0)} f + \dots \end{aligned}$$

$$|E(x, y)| \leq \frac{M}{2} (|x - x_0| + |y - y_0|)^2 \quad \text{where} \quad |f_{xx}|, |f_{xy}|, |f_{yy}| \leq M$$

$$M = \iint_R \delta(x, y) dA \quad M_x = \iint_R y \delta(x, y) dA \quad M_y = \iint_R x \delta(x, y) dA$$

$$(\bar{x}, \bar{y}) = \left( \frac{M_y}{M}, \frac{M_x}{M} \right)$$

$$I_x = \iint_R y^2 \delta(x, y) dA \quad I_y = \iint_R x^2 \delta(x, y) dA \quad I_0 = \iint_R (x^2 + y^2) \delta(x, y) dA$$

$$R_x = \sqrt{I_x/M} \quad R_y = \sqrt{I_y/M}$$

$$\begin{array}{lll} x = r \cos(\theta) & r = \rho \sin(\phi) & x = \rho \sin(\phi) \cos(\theta) \\ y = r \sin(\theta) & z = \rho \cos(\phi) & y = \rho \sin(\phi) \sin(\theta) \\ z = z & \theta = \theta & z = \rho \cos(\phi) \end{array}$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}(y/x) \quad \phi = \tan^{-1}(r/z)$$

$$dV = dx dy dz = r dr d\theta dz = \rho^2 \sin(\phi) d\rho d\phi d\theta$$

— OVER —

$$\iiint_V f(x, y, z) dx dy dz = \iiint_{V'} f(x(u, v, w), y(u, v, w), z(u, v, w)) |J(u, v, w)| du dv dw$$

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\text{Work: } \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy$$

$$\text{Flux: } \int_C \mathbf{F} \cdot \mathbf{n} ds = \int_C M dy - N dx$$

$$\text{Vector identities: } \nabla \times (\nabla f) = \mathbf{0}, \quad \nabla \cdot (\nabla \times \mathbf{F}) = 0$$

$$\text{Conservative field: } \nabla \times \mathbf{F} = \mathbf{0} \Leftrightarrow \begin{cases} \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \\ \frac{\partial P}{\partial z} = \frac{\partial M}{\partial x} \\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \end{cases}$$

$$\text{For a surface given by } g(x, y, z) = 0: \quad \iint_S f(x, y, z) d\sigma = \iint_R f(x, y, z) \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} dA$$

$$\text{Flux of } \mathbf{F} \text{ through surface given by } g(x, y, z) = 0: \quad \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_R \frac{\pm \mathbf{F} \cdot \nabla g}{|\nabla g \cdot \mathbf{p}|} dA$$

**Green's Theorem:**

$$\text{Circulation} = \oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_R (\nabla \times \mathbf{F}) \cdot \hat{k} dA = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$\text{Flux} = \oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_R \nabla \cdot \mathbf{F} dA = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

**Stokes's Theorem:**

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C M dx + N dy + P dz = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma$$

**Divergence Theorem of Gauss:**

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F} dV$$