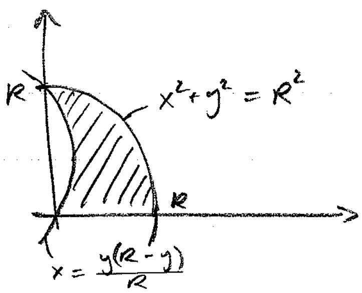
1 (a) FALSE (b) FALSE (c) TRUE (d) FALSE (e) FALSE

2 (a) $\vec{F} = -\left[\frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath} + \frac{\partial T}{\partial z}\hat{k}\right] = \boxed{yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}}$

(b)



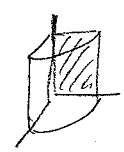
(c) Outward flux $= \iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_{\text{shark fin}} \vec{F}\cdot\hat{k}\,dA$ ($d\sigma = dA$ because surface is flat)

$$= \iint xy\,dA = \boxed{\int_0^R \int_{\frac{y(R-y)}{R}}^{\sqrt{R^2-y^2}} xy\,dx\,dy}$$

(d) Outward flux $= \iint_{\text{shark fin}} \vec{F}\cdot(-k)\,dA = \iint -xy\,dA = -\iint xy\,dA$

= negative of integral in (b).

(e) $\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_{\text{rectangle}} \frac{\vec{F}\cdot\nabla g}{|\nabla g\cdot\hat{p}|}\,dA$



where $g = x^2+y^2-R^2$ & $\hat{p} = \hat{\imath}$. The rectangle is $\begin{cases} 0 \le y \le R \\ 0 \le z \le H \end{cases}$

$\nabla g = 2x\hat{\imath} + 2y\hat{\jmath} \Rightarrow \vec{F}\cdot\nabla g = 2xyz + 2xyz = 4xyz$

$|\nabla g\cdot\hat{p}| = |2x| = 2x$

$\Rightarrow \text{Flux} = \int_0^H \int_0^R 2yz\,dy\,dz = \int_0^H [y^2z]_0^R\,dz = \int_0^H R^2 z\,dz$

$= R^2\left[\frac{z^2}{2}\right]_0^H = \boxed{\frac{R^2H^2}{2}}$

2 (f) The entire surface is closed, so by Gauss Divergence Thm

$$\iint_S \vec{F}\cdot\hat{n}\, d\sigma = \iiint_D \nabla\cdot\vec{F}\, dV$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0 \Rightarrow \iiint_D \nabla\cdot\vec{F}\, dV = 0$$

ie no net flux $\Rightarrow \iint_{x^2+y^2=R^2} + \iint_{y=0} + \iint_{x=\frac{y(R-y)}{R}} + \iint_{z=0} + \iint_{z=H} = 0$

From (d) $\iint_{z=0} + \iint_{z=H} = 0$ so $\iint_{y=0} + \iint_{x=\frac{y(\,)}{R}} = -\iint_{x^2+y^2=R^2} = \boxed{\frac{-R^2H^2}{2}}$

from (e)

3 Need to find max/min of density $p = e^{x^2+2y^2}$ subject to the constraint $2x^2+2xy+5y^2 = 9$.

Lagrange: $\nabla p = 2xe^{x^2+2y^2}\,\hat{\imath} + 4ye^{x^2+2y^2}$

$\nabla g = (4x+2y)\,\hat{\imath} + (2x+10y)\,\hat{\jmath}$

$$2xe^{x^2+2y^2} = 2\lambda(2x+y) \Rightarrow e^{x^2+2y^2} = \frac{\lambda(2x+y)}{x} \quad \text{or } x=0$$

$$4ye^{x^2+2y^2} = 2\lambda(x+5y) \Rightarrow e^{x^2+2y^2} = \frac{\lambda(x+5y)}{2y} \quad \text{or } y=0$$

$$2x^2+2xy+5y^2 = 9$$

If $x=0$, $0 = 2\lambda(y) \Rightarrow y=0$ or $\lambda=0$

$x=y=0$ contradicts the constraint. $\lambda=0 \Rightarrow 4ye^{-}=0 \Rightarrow y=0$ which, again, is a contradiction.

Similarly $y=0 \Rightarrow x=0$ ↯ So $x\neq0,\ y\neq0$

$$e^{x^2+2y^2} = \frac{\lambda(2x+y)}{x} = \frac{\lambda(x+5y)}{2y} \Rightarrow 2\lambda y(2x+y) = \lambda x(x+5y)$$

$$\Rightarrow \lambda = 0 \quad \text{OR} \quad 2y(2x+y) = x(x+5y)$$

$\lambda = 0 \Rightarrow x = y = 0$ which is a contradiction

So
$$4xy + 2y^2 = x^2 + 5xy \Rightarrow x^2 + xy = 2y^2$$
$$2x^2 + 2xy + 5y^2 = 9 \Rightarrow 2(x^2+xy) + 5y^2 = 9$$

$$\Rightarrow 2(2y^2) + 5y^2 = 9y^2 = 9 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$$

$$y = 1 \Rightarrow x^2 + x = 2 \Rightarrow x^2 + x - 2 = (x+2)(x-1) = 0 \Rightarrow x = 1, -2$$
$$y = -1 \Rightarrow x^2 - x = 2 \Rightarrow x^2 - x - 2 = (x-2)(x+1) = 0 \Rightarrow x = -1, 2$$

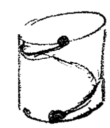
$\Rightarrow$ Possibilities are $(x,y) = (\pm 1, \pm 1)$ & $(\pm 2, \mp 1)$

$$p(\pm 1, \pm 1) = e^{1+2} = e^3 \qquad p(\pm 2, \mp 1) = e^{4+2} = e^6$$

$\Rightarrow$ Best places to hunt are $(x,y) = (\pm 2, \mp 1)$
Worst " " " " $(x,y) = (\pm 1, \pm 1)$

4. $\vec{F} = (2xy + z^3)\hat{\imath} + (x^2 - \sin y)\hat{\jmath} + (3xz^2)\hat{k}$

Work done from $(R,0,0)$ to $(R,0,H)$ on helical path



Is $\vec{F}$ conservative?

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy + z^3 & x^2 - \sin y & 3xz^2 \end{vmatrix}$$

$$= \left[\tfrac{\partial}{\partial y}(3xz^2) - \tfrac{\partial}{\partial z}(x^2 - \sin y) \right]\hat{\imath} + \left[\tfrac{\partial}{\partial z}(2xy + z^3) - \tfrac{\partial}{\partial x}(3xz^2) \right]\hat{\jmath} + \left[\tfrac{\partial}{\partial x}(x^2 - \sin y) - \tfrac{\partial}{\partial y}(2xy + z^3) \right]\hat{k}$$

$$= (0-0)\hat{\imath} + (3z^2 - 3z^2)\hat{\jmath} + (2x - 2x)\hat{k} = \vec{0}$$

So, yes, $\vec{F}$ is conservative $\Rightarrow \int_C \vec{F}\cdot d\vec{r}$ is path independent.

Define $\vec{r}(t) = R\hat{\imath} + 0\hat{\jmath} + t\hat{k}$ for $t \in [0, H]$

This path also goes from $(R,0,0)$ to $(R,0,H)$ $\Rightarrow$ same work done.

$$\Rightarrow \int_C \vec{F}.d\vec{r} = \int_0^H \vec{F}\cdot(0\hat{\imath} + 0\hat{\jmath} + \hat{k})\, dt = \int_0^H 3xz^2\, dt$$

$$= \int_0^H 3Rt^2\, dt = R\left[t^3\right]_0^H = \boxed{RH^3}$$

[Alternatively, find potential function $f(x,y,z) = x^2y + \cos y + xz^3 + C$

Then $\int_C \vec{F}\cdot d\vec{r} = f(R,0,H) - f(R,0,0)$

$$= [R^2 \times 0 + \cos(0) + RH^3 + C] - [R^2 . 0 + \cos(0) + R . 0^3 + C]$$

$$= (1 + RH^3 + C) - (1 + C)$$

$$= RH^3$$]

5. $T = \frac{3}{8}x^2 + yz \quad \Rightarrow \nabla T = \frac{3}{4}x\,\hat{\imath} + z\,\hat{\jmath} + y\,\hat{k}$

$\vec{r}(t) = 2t\,\hat{\imath} + 3t\,\hat{\jmath} + t^2\hat{k} \quad \Rightarrow$ @ $(4,6,4)$ when $t = 2$.

a) $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} = \nabla T \cdot \frac{d\vec{r}}{dt}$

$\nabla T(4,6,4) = 3\hat{\imath} + 4\hat{\jmath} + 6\hat{k}$

$\frac{d\vec{r}}{dt} = 2\hat{\imath} + 3\hat{\jmath} + 2t\,\hat{k} \quad \Rightarrow \left.\frac{d\vec{r}}{dt}\right|_{t=2} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$

$\Rightarrow \frac{dT}{dt} = (3\hat{\imath} + 4\hat{\jmath} + 6\hat{k})\cdot(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) = 6 + 12 + 24 = \boxed{42}$ °C/min

b) $\frac{dT}{ds} = \frac{dT}{dt}\frac{dt}{ds} = \frac{dT}{dt}\frac{1}{|\vec{v}|} \qquad |\vec{v}| = |2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}| = \sqrt{4+9+16} = \sqrt{29}$

$\Rightarrow \frac{dT}{ds} = \boxed{\frac{42}{\sqrt{29}}}$ °C/m

c) Direction of fastest decrease of T is $-\nabla T = -3\hat{\imath} - 4\hat{\jmath} - 6\hat{k}$

Rate of change of T in that direction is $D_{\hat{u}}T = \nabla T \cdot \hat{u}$

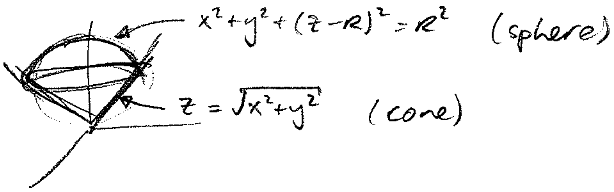
$$= \nabla T \cdot \frac{-\nabla T}{|\nabla T|}$$

$$= \frac{-(\nabla T)\cdot(\nabla T)}{|\nabla T|} = \frac{-|\nabla T|^2}{|\nabla T|}$$

$\Rightarrow$ Max rate of decrease of T is $-|\nabla T| = -\sqrt{9+16+36} = -\sqrt{61}$

$\Delta T \approx$ (Rate of change) $\times$ (distance traveled) $= (-\sqrt{61})(0.1) = \frac{-\sqrt{61}}{10} = \boxed{-\sqrt{0.61}}$

6



a) Intersection is $x^2+y^2 = R^2-(z-R)^2$

$$= z^2$$

$$\Rightarrow z^2 = R^2-(z-R)^2 = R^2-(z^2-2zR+R^2) = 2zR - z^2$$

$$\Rightarrow \cancel{2}z^2 = \cancel{2}zR \Rightarrow z \neq 0 \quad \text{OR} \quad \boxed{\begin{array}{l} z=R \\ \Rightarrow x^2+y^2=R^2 \end{array}}$$

So C is circle in $z=R$ plane

$$\Rightarrow \boxed{\vec{r}(t) = R\cos t\,\hat{\imath} + R\sin t\,\hat{\jmath} + R\hat{k}}$$

b) Circulation $= \oint_C \vec{F}\cdot d\vec{r} = \int_C (x^2\hat{\imath}+y^2\hat{\jmath}+xz\hat{k})\cdot(-R\sin t\,\hat{\imath}+R\cos t\,\hat{\jmath}+0\hat{k})\,dt$

$$= \int_0^{2\pi} -R^3\cos^2 t\sin t + R^3\sin^2 t\cos t \, dt = R^3\int_0^{2\pi} \sin^2 t\cos t - \cos^2 t\sin t \, dt$$

$$= R^3\left[\tfrac{1}{3}\sin^3(t) + \tfrac{1}{3}\cos^3(t)\right]_0^{2\pi} = \boxed{0}$$

c) $\vec{F}$ is continuously differentiable so, using Stokes's Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,d\sigma$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & xz \end{vmatrix} = (0-0)\hat{\imath} + (0-z)\hat{\jmath} + (0-0)\hat{k} = -z\hat{\jmath}$$

Taking S to be the plane $z=R$, $\hat{n}=\hat{k} \Rightarrow \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,d\sigma$

$$= \iint_D 0\,dA = 0$$

7. Mass $= \iint_S \rho(x,y,z)\, d\sigma$ where S is the surface $g(x,y,z) = 0$ where $g(x,y,z) = z - x^2 - y^2$.

$$\Rightarrow M = \iint_R \frac{3xy}{\sqrt{1+4(x^2+y^2)}} \frac{|\nabla g|}{|\nabla g \cdot \hat{p}|}\, dA$$

Taking $\hat{p} = \hat{k}$, R is region bounded by $x = y^2$, $x = y^2/3$, $y = x^2/2$, $y = x^2/5$

$\nabla g = -2x\,\hat{\imath} - 2y\,\hat{\jmath} + \hat{k}$ $\qquad \nabla g \cdot \hat{k} = 1$

$|\nabla g| = \sqrt{4x^2 + 4y^2 + 1}$

$$\Rightarrow M = \iint_R \frac{3xy}{\sqrt{}} \frac{\sqrt{}}{|1|}\, dA = \iint_R 3xy\, dA$$

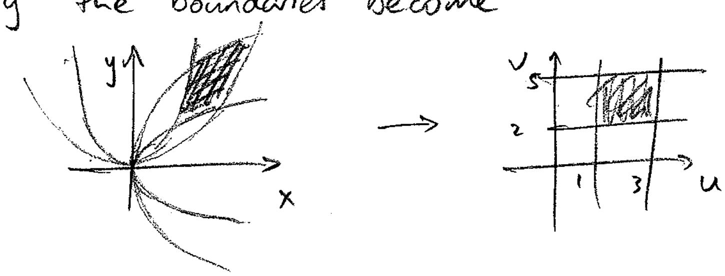
Using $u = y^2/x$ $\quad v = x^2/y$ the boundaries become

$x = y^2 \iff u = 1$

$x = y^2/3 \iff u = 3$

$y = x^2/2 \iff v = 2$

$y = x^2/5 \iff v = 5$



The integrand $3xy = 3u^{1/3}v^{2/3}u^{2/3}v^{1/3} = 3uv$

The Jacobian is $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}u^{1/3}v^{-1/3} \\ \frac{2}{3}u^{-1/3}v^{1/3} & \frac{1}{3}u^{2/3}v^{-2/3} \end{vmatrix} = \frac{1}{9} - \frac{4}{9} = \frac{-3}{9}$
$= \frac{-1}{3}$

$$\Rightarrow M = \iint_{R'} 3uv\left|\tfrac{-1}{3}\right| du\,dv = \int_2^5 \int_1^3 uv\, du\, dv = \int_2^5 \left[\frac{u^2 v}{2}\right]_1^3 dv$$

$$= \int_2^5 \frac{8v}{2}\, dv = \int_2^5 4v\, dv = \left[2v^2\right]_2^5 = 2(21) = \boxed{42}\text{ kg}$$