
INSTRUCTIONS: Computers, calculators, books, notes, flaming monkeys, *etc.* are not permitted. Some (possibly) useful formulae are attached. Write your name, your instructor's name and your recitation section (and/or TA's name) on the front of your bluebook. Work all problems. Start each problem on a **new page**. Unless otherwise stated, show **all** your work clearly and box your final answer; a correct answer with incorrect or no supporting work will **receive no credit**, while an incorrect answer with relevant work may receive partial credit.

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1. Decide whether the following statements are (always) TRUE or FALSE (not always true). You **must** write the full word — "T" or "F" alone will NOT be graded! You do not need to show working for this problem.

   (a) The integral $\int_0^2 \int_{x+2}^4 xy^2 \sin(y)\, dydx$ gives the moment of inertia around the x-axis of a triangular shape with density $\delta(x, y) = x \sin(y)$.

   (b) If (x^*, y^*) is a critical point of f and $f_{yy}|_{(x^*,y^*)} < 0$ and $f_{xy}|_{(x^*,y^*)} < 0$, then (x^*, y^*) is a maximum of f.

   (c) If there is a direction $\mathbf{u}$ such that $D_{\mathbf{u}} f|_{(x^*,y^*)} = 0$, then (x^*, y^*) is a max or min of f.

   (d) If $R = \{(x, y) : (x - 3)^2 + (y + 1)^2 \le 4\}$ and $I = \iint_R \sin(x^2 - y^7) \sin(\pi x^5 y)\, dA$ then $-4\pi \le I \le 4\pi$.

   (e) If f and all its partial derivatives are continuous everywhere, then (in general) f has five distinct fourth-order partial derivatives.

2. Water is an incompressible fluid for which the pressure P, velocity v, density ρ, and height of the fluid z above some reference elevation are related by Bernoulli's equation

$$P + \frac{1}{2}\rho v^2 + \rho g z = \text{constant}$$

   where g is the constant acceleration due to gravity. Assume that, under some circumstance, the density of the water is 1000 kg/m^3, the velocity is 200 m/s, and the height is 1 m. Approximately what is the change in the pressure of the water if the density is <u>in</u>creased by 10 kg/m^3, the velocity is <u>de</u>creased by 1 m/s, and the elevation is <u>in</u>creased by 0.1 m? Assume that $g = 10$ m/s^2.

3. In a quantum mechanical system consisting of an electron contained within a right-circular cylinder, the minimum kinetic energy that the electron can have is given by

$$E = \frac{\hbar}{2m}\left(\frac{k}{R^2} + \frac{\pi^2}{H^2}\right) = C\left(\frac{k}{R^2} + \frac{\pi^2}{H^2}\right)$$

   where R and H are the radius and height, respectively, of the cylinder, and $\hbar$, k and m (and, therefore, C) are all constants. Find the <u>ratio</u> R/H that gives the minimum energy E for a cylinder of fixed volume V. [Hint: start as if you were going to find R and H explicitly in terms of V, but as you proceed remember that we only need R/H at the end.]

4. Consider the function $f(x, y) = x^2 + y^2 - 2\cos(y)$.
   (a) Find all local maxima, minima and saddle points (and state clearly which type they are).
   (b) Are any of the maxima or minima global max/mins? [Hint: look at the values of f as x and $y \to \pm\infty$.]
   (c) Sketch the domain $D = \{(x, y) : x^2 + (y - 1)^2 \leq 4\}$.
   (d) Find the global max and min of f over the domain D.

5. The temperature distribution of the antarctic ice is given by a function $T(x, y)$. A penguin walks along a 2-D path $\mathbf{r}(t)$. At time $t = 2.3$, the penguin is at the point $(-1, 4)$ and has velocity $\mathbf{v}(2.3) = \hat{\imath} - 2\hat{\jmath}$ km/hr. You also know that $T(-1, 4) = -40°$ and $\nabla T|_{(-1,4)} = 3\hat{\imath} + 2\hat{\jmath}$.
   (a) Calculate the rate of change of temperature with respect to time, in °/hr, that the penguin is experiencing at $t = 2.3$, or explain why it cannot be calculated from the information given.
   (b) Calculate the rate of change of temperature with respect to distance, in °/km, that the penguin is experiencing at $t = 2.3$, or explain why it cannot be calculated from the information given.
   (c) The penguin has just realized that it's very cold in the antarctic and so would like to change direction and go somewhere warmer. What direction should the penguin go to warm up the fastest?
   (d) On further reflection, the penguin has decided that $-40°$ is a very good temperature (it's a very well-educated penguin and knows that -40 is where the Fahrenheit and Celsius scales align). What direction should the penguin go to keep the temperature constant?

— Useful and interesting formulae —

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\vec{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\vec{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(u)|\,du \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{[\dot{x}^2+\dot{y}^2]^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \qquad \text{where} \qquad a_T = \frac{d}{dt}|\mathbf{v}|, \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

$$\frac{df}{ds} = D_{\mathbf{u}}f = (\nabla f)\cdot\mathbf{u}$$

$$\text{Discriminant: } \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$$

$$\nabla f = \lambda\nabla g \qquad g = 0$$

$$\begin{aligned} f(x,y) \quad &= \quad f(0,0) + \left(\left.\frac{\partial f}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial f}{\partial y}\right|_{(0,0)} y\right) \\ &\quad + \frac{1}{2}\left(\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} x^2 + 2\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(0,0)} xy + \left.\frac{\partial f}{\partial y^2}\right|_{(0,0)} y^2\right) \\ &\quad + \ldots + \frac{1}{n!}\left.\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n\right|_{(0,0)} f + \ldots \end{aligned}$$

$$|E(x,y)| \le \frac{M}{2}\left(|x-x_0| + |y-y_0|\right)^2 \qquad \text{where} \qquad |f_{xx}|, |f_{xy}|, |f_{yy}| \le M$$

$$M = \iint_R \delta(x,y)\,dA \qquad M_x = \iint_R y\delta(x,y)\,dA \qquad M_y = \iint_R x\delta(x,y)\,dA$$

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

$$I_x = \iint_R y^2\delta(x,y)\,dA \qquad I_y = \iint_R x^2\delta(x,y)\,dA \qquad I_0 = \iint_R (x^2+y^2)\delta(x,y)\,dA$$

$$R_x = \sqrt{I_x/M} \qquad R_y = \sqrt{I_y/M}$$