
INSTRUCTIONS: Computers, calculators, books, notes, flailing monkeys, *etc.* are not permitted. Some (possibly) useful formulae are attached. Write your name, your instructor's name and your recitation section (and/or TA's name) on the front of your bluebook. Work all problems. Start each problem on a **new page**. Unless otherwise stated, show **all** your work clearly and box your final answer; a correct answer with incorrect or no supporting work will **receive no credit**, while an incorrect answer with relevant work may receive partial credit.

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1. (20 points)

   (a) Find the constants a, b, c so that the vector field defined by $\mathbf{F} = (x+2y+az)\hat{\imath} + (bx-3y-z)\hat{\jmath} + (4x+cy+2z)\hat{k}$ is conservative.

   (b) Is $\mathbf{F} = xy\hat{\imath} - z\hat{\jmath} + x^2\hat{k}$ a conservative vector field? If so, find a scalar function f such that $\mathbf{F} = \nabla f$.

   (c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}$ is the vector field in (b) and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$. You may leave your answer as a sum of fractions.

2. (25 points) Geoff, the proprietor of Land O' Lake Ice Cream Company, is testing a new machine for pouring ice cream into the cartons for sale. Given a set of axes in which the carton is the cuboid region $\{(x,y,z) : 0 \leq x \leq 4, 0 \leq y \leq 2, 0 \leq z \leq 2\}$, the machine is pouring the ice cream in such a way that the distribution of Crunchy Frog Bits$^{\text{TM}}$ in Geoff's revolutionary frog-flavored ice cream is given by the equation $\delta(x,y,z) = 2 + z^2$ [bits/in^3]. Megan, one of Geoff's intrepid taste-testers, has scooped out a lump of ice cream from the top of the carton in the shape of the hemisphere $\{(x,y,z) : (x-2)^2 + (y-1)^2 + (z-2)^2 \leq 1, z \leq 2\}$.

   (a) Sketch the ice-cream carton with the hemisphere removed. Label your axes!

   (b) Calculate the total number of frog bits in the whole carton (before the scoop is removed).

   (c) Show that the volume element dV is unchanged under translation of coordinates — *i.e.* $u = x - x_0$, $v = y - y_0$, $w = z - z_0$, where x_0, y_0, z_0 are constants.

   (d) Use the result in (c), and another coordinate change, to set up the integral necessary to calculate the total number of frog bits in Megan's scoop using the most natural set of coordinates. (You do NOT need to evaluate the integral.)

3. (20 points) Calculate the integral of $f(x,y,z) = 2z$ over the wedge in the first octant cut from the solid cylinder $y^2 + z^2 \leq 1$ by the plane $y = x$.

4. (15 points) Convert $\displaystyle\int_{-1}^{1}\int_{1}^{\sqrt{2-y^2}} \frac{x}{x^2+y^2}\, dx dy$ into polar coordinates (but do not evaluate it).

5. (20 points) Integrating over an ellipse $R = \{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\}$ is difficult, even in polar coordinates. However, the change of coordinates $x = au\cos(v)$, $y = bu\sin(v)$, where $u \ge 0$ and $0 \le v \le 2\pi$, can make the problem feasible.

   (a) Write the integral $\iint_R f(x,y)\,dA$ in the new coordinates (u,v).

   (b) Use your result from (a) to find the area of R.

   (c) Verify that your result in (b) makes sense when $a = b$.

— Useful and interesting formulae —

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\vec{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\vec{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(u)|\,du \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{[\dot{x}^2+\dot{y}^2]^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \qquad \text{where} \qquad a_T = \frac{d}{dt}|\mathbf{v}|, \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

$$\frac{df}{ds} = D_{\mathbf{u}}f = (\nabla f)\cdot\mathbf{u}$$

$$\text{Discriminant: } \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$$

$$\nabla f = \lambda\nabla g \qquad g = 0$$

$$\begin{aligned} f(x,y) &= f(0,0) + \left(\left.\frac{\partial f}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial f}{\partial y}\right|_{(0,0)} y\right) \\ &+ \frac{1}{2}\left(\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} x^2 + 2\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(0,0)} xy + \left.\frac{\partial f}{\partial y^2}\right|_{(0,0)} y^2\right) \\ &+ \ldots + \frac{1}{n!}\left.\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n\right|_{(0,0)} f + \ldots \end{aligned}$$

$$|E(x,y)| \le \frac{M}{2}\left(|x-x_0| + |y-y_0|\right)^2 \qquad \text{where} \qquad |f_{xx}|, |f_{xy}|, |f_{yy}| \le M$$

$$M = \iint_R \delta(x,y)\,dA \qquad M_x = \iint_R y\delta(x,y)\,dA \qquad M_y = \iint_R x\delta(x,y)\,dA$$

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

$$I_x = \iint_R y^2\delta(x,y)\,dA \qquad I_y = \iint_R x^2\delta(x,y)\,dA \qquad I_0 = \iint_R (x^2+y^2)\delta(x,y)\,dA$$

$$R_x = \sqrt{I_x/M} \qquad R_y = \sqrt{I_y/M}$$

$$\begin{array}{rclcrclcrcl} x &=& r\cos(\theta) && r &=& \rho\sin(\phi) && x &=& \rho\sin(\phi)\cos(\theta) \\ y &=& r\sin(\theta) && z &=& \rho\cos(\phi) && y &=& \rho\sin(\phi)\sin(\theta) \\ z &=& z && \theta &=& \theta && z &=& \rho\cos(\phi) \end{array}$$

$$\rho = \sqrt{x^2+y^2+z^2} \qquad r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}(y/x) \qquad \phi = \tan^{-1}(r/z)$$

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

$$\begin{aligned} \iiint_V f(x,y,z)\,dx\,dy\,dz &= \iiint_{V'} f(x(u,v,w),y(u,v,w),z(u,v,w))|J(u,v,w)|\,du\,dv\,dw \\ J(u,v,w) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \end{aligned}$$

$$\begin{aligned} \text{Work: } \int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy \\ \text{Flux: } \int_C \mathbf{F}\cdot\mathbf{n}\,ds &= \int_C M\,dy - N\,dx \end{aligned}$$

$$\text{Conservative field: } \nabla\times\mathbf{F} = \mathbf{0} \Leftrightarrow \begin{cases} \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \\ \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z} \\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \end{cases}$$